Note about viewing and printing:

For the best viewing experience, download and save this book to your computer or tablet and open it from there. This page is here to help with two-page viewing on your computer or tablet screen. It ensures even-numbered pages are displayed on the left and odd-numbered pages on the right. We expect on-screen viewing will be the primary way people will use this book. We tested it with freeware Adobe Acrobat Reader (`https://get.adobe.com/reader`) on various devices. If you want to print the book two-sided, then it is best to exclude this page.

From the reviews:

This book looks amazing! It will provide tremendous value to students preparing for the $F = ma$ competition. The solutions are very detailed, much more detailed and clearer than any solutions (usually partial or a sketch) available elsewhere. I can envision individuals and groups, such as physics clubs in schools, working on the $F = ma$ problems and comparing their solutions to this books insightful solutions. I believe they will learn a lot this way! Also, oftentimes, physics is not taught with multiple approaches, and this book shows students that there are indeed multiple ways to approach a problem.

Charles Liang
AwesomeMath Summer Program Staff
PhD student, University of Chicago

This is the latest version as of **December 25, 2018**.

Branislav Kisačanin Eric K. Zhang

DRAFT VERSION

$F = ma$ Contests

2011-2018 Solutions Manual

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This book was typeset in LaTeX by BK and EKZ.
Cover design by BK.
Front: Likeness of Sir Isaac Newton (1642-1727),
who first discovered the formula $F = ma$.

We would like to dedicate this book
to our physics teachers:

Miodrag Zatezalo, Gena Litričin, Darko Kapor

BK

&

Jerry Grizzle

EKZ

I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Sir Isaac Newton

Amazing modesty from someone who indebted science with discoveries of Laws of Motion and Law of Gravity, invention of differential and integral calculus, and so much more! For a memorable tribute to Newton, we recommend Dr. Neil deGrasse Tyson's interview *My Man, Sir Isaac Newton*:

`https://www.youtube.com/watch?v=danYFxGnFxQ`.

Contents

Preface

In this book we present detailed solutions of problems that were given in annual $F = ma$ physics competitions between 2011 and 2018. We wrote it with a goal of bridging a significant gap in existing literature for students preparing for physics competitions. In this book many problems are solved in more than one way, demonstrating that the same problem can be viewed from different angles and solved using, at times, very different approaches. We can tell you right away that this book is not a textbook - it does not aim to teach physics concepts, but we did our best to make it a great resource for competition preparation, using our experiences as an instructor of competitive physics classes at the AwesomeMath Academy (BK) and a two-time gold medalist at US National Physics Olympiads (EKZ).

Physics competitions. The $F = ma$ competition is organized every year in January by the AAPT (American Association of Physics Teachers). It is the first level of high school physics competitions, leading to the USAPhO (United States National Physics Olympiad) as the next level, and eventually to the IPhO (International Physics Olympiad). Here is some IPhO trivia: the US Physics Teams have participated in IPhO since 1986; the first IPhO was held in 1967 in Warsaw, Poland.

$\boldsymbol{F = ma}$ competitions. As you will see in this book, $F = ma$ competitions cover only problems in classical mechanics, and the math required to solve them is, for the most part, at the level of precalculus. Only rarely do we see problems requiring calculus. This, however, does not mean the problems are easy, on the contrary, you will see that the exams are a mix of problems with a rather big range of difficulty, from trivial to extremely hard.

Problem difficulty levels. Because of the big range in problem difficulty, we found it useful in our teaching practice to categorize these problems in three levels, corresponding to three levels of prior knowledge that most students entering our classes bring with them:

- Level 1: Students with no prior exposure to physics problem solving will find the problems we call "easy" (marked with a single star ($\star$)) interesting and moderately challenging.

- Level 2: Students who completed a pre-AP high school physics class and already had some exposure to physics problem solving, will find the problems we call "medium hard" (marked with $\star\star$) enjoyable and challenging. Once comfortable with "medium hard" problems, students have a good chance of qualifying to USAPhO.
- Level 3: Students who completed an AP Physics class and are proficient with "easy" and "medium hard" problems, should attempt "very hard" ($\star\star\star$) problems as a part of USAPhO preparations.

While any such ranking is necessarily subjective, we found this classification to be very helpful in addressing the big range of problem difficulties encountered in $F = ma$ exams. This classification is additionally useful because each problem, regardless of difficulty, carries one point. Therefore, being able to recognize the level of problem difficulty, right there, in the competition, is another useful skill that students can develop based on this ranking.

This classification is also helpful with respect to a common student question: "Which problems should I focus on in the competition?" Another common student question is "How many points are needed to qualify to USAPhO?" The following table shows the minimum number of points that was needed to qualify to the USAPhO in each year, as well as how many of the 25 problems each year were "easy" ($\star$), "medium hard" ($\star\star$), and "very hard" ($\star\star\star$). You can see that students who did most of "easy" and "medium hard" problems qualified for the USAPhO.

Year	Minimum Passing Score	$\star$	$\star\star$	$\star\star\star$
2011	unknown	13	7	5
2012	15.50	8	13	4
2013	12.25	14	8	3
2014	12.50	10	8	7
2015$^\times$	18	12	9	4
2016$^\times$	14	10	10	5
2017$^\times$	16	9	12	4
2018-A$^\times$	14	8	13	4
2018-B$^\times$	15	10	11	4

$^\times$ denotes years in which points were not deducted for incorrect answers.

Finally, let us mention that this difficulty ranking also influenced the style we used to write our solutions: solutions for "easy" problems are accessible to Level 1 students, solutions for "medium hard" problems are meant for Level 2 students, while solutions for "very hard" problems are aimed at Level 3 students.

In each chapter of the book, between exam problems and solutions, you will find another kind of useful tables. For each problem in that chapter, these tables provide the correct answer choice, problem difficulty level (denoted by stars), and the area(s) of physics corresponding to the problem. With this information, students can navigate the book faster. For example, if on a particular day students want to focus on, say, "medium hard" problems using Archimedes' Principle, these tables will be very useful.

About using this book. The goals and purpose of this book are rather narrow, so let us first make a few important points about what it is and what it is not:

- This book does not teach new physics concepts, i.e., it is not a textbook. See Additional Resources at the end of the book for a few textbook recommendations.
- This book is a collection of contest problems and is therefore not the best problem set to *learn* physics problem solving techniques. That is why in our competitive physics classes we use a completely different problem set that is geared towards *learning* physics problem solving techniques.
- This book is a great resource to *practice* and *improve* physics problem solving skills. Indeed, that is how we use it in our classes, as supplementary material.
- This book provides a lot of other information useful for successful competition preparations: problem ranking and topics, correct answer choices, detailed solutions, and links to other resources.

With all this in mind, here are a few thoughts on how to best use this book:

- Early on, pick two recent exams, for example 2017 and 2018-B, that you will not open at first. If they are left untouched, you can use them later to take a mock-up exam, with limited time and previously unseen problems.
- Use the tables of topics and difficulty levels to try a few problems in order to get a feel for the types of problems given at $F = ma$ exams and their range of difficulty.
- Identify areas of physics in which your knowledge is not at the "I can pass the $F = ma$ exam" level; if the gaps are small, the solutions in this book will likely be sufficient to fill them. If gaps are bigger, textbooks and other resources will be helpful.
- Figure out for yourself how much time you should spend on "easy" vs. "medium hard" problems and whether or not you can afford to try some of the "very hard" problems during the competition.
- Take a test that you didn't open before and try to do it with time constraint (75 minutes is the time specified by the contest rules).

More about this book. While writing this book, we made sure that derivations and explanations are complete and for the most part self-contained. To make the material more interesting, we often inserted external links to YouTube, Wikipedia, and other educational places on the Web. At the end of the book we provide several appendices and list additional resources that we consider useful for students preparing for physics competitions and their teachers.

Acknowledgments. We would like to thank the people who wholeheartedly encouraged us to write this book: Dr. Titu Andreescu at the AwesomeMath Academy and Drs. Robert Hilborn and Juan Burciaga at the AAPT. Our families deserve a big thanks for offering the support we needed to focus on this project. We would also like to thank Marko Matić and Mason Fang, Physics TAs at the AwesomeMath Academy, and Charles Liang, a TA at the AwesomeMath Summer Programs, for their initial reviews of the material. We also appreciate the work done by the anonymous reviewers selected by the AAPT, for their kind comments and valuable suggestions. Several of our AwesomeMath Academy students gave valuable suggestions: Bill Wang, Sophia Barnes, and Dennis Li. We sincerely thank the AAPT sponsors, whose generous donations helped make this book available as a free download. Last, but certainly not least, we would like to acknowledge members of the US Physics Team and their coaches over the years for creating problems given in $F = ma$ competitions, and therefore problems presented and solved in this book.

Contact us! Having done all this work for you, our reader, we would appreciate hearing from you. Have you found a mistake that we can correct, invented a new way to use this book, or have something nice to say about the book? Please share with us over e-mail!

Branislav Kisačanin
`b.kisacanin@yahoo.com`

Eric K. Zhang
`ekzhang1@gmail.com`

November 20, 2018 — Plano, TX

Year 2011

$\boldsymbol{F = ma}$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2011

2011 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

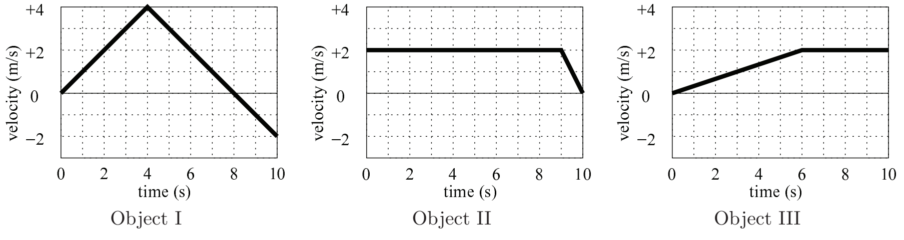
- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers will result in a deduction of $\frac{1}{4}$ point. There is no penalty for leaving an answer blank.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2011.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

1. A cyclist travels at a constant speed of 22.0 km/hr except for a 20 minute stop. The cyclist's average speed was 17.5 km/hr. How far did the cyclist travel?

    (A) 28.5 km
    (B) 30.3 km
    (C) 31.2 km
    (D) 36.5 km
    (E) 38.9 km

Questions 2 to 4 refer to the three graphs below which show velocity of three objects as a function of time. Each object is moving only in one dimension.



2. Rank the *magnitudes* of the average acceleration during the ten second interval.

    (A) I > II > III
    (B) II > I > III
    (C) III > II > I
    (D) I > II = III
    (E) I = II = III

3. Rank the *magnitudes* of the maximum velocity achieved during the ten second interval.

    (A) I > II > III
    (B) II > I > III
    (C) III > II > I
    (D) I > II = III
    (E) I = II = III

4. Rank the *magnitudes* of the *distance* traveled during the ten second interval.

    (A) I > II > III
    (B) II > I > III
    (C) III > II > I
    (D) I = II > III
    (E) I = II = III

5. A crude approximation is that the Earth travels in a circular orbit about the Sun at constant speed, at a distance of 150,000,000 km from the Sun. Which of the following is the closest for the acceleration of the Earth in this orbit?

    (A) exactly 0 m/s^2
    (B) 0.006 m/s^2
    (C) 0.6 m/s^2
    (D) 6 m/s^2
    (E) 10 m/s^2

6. A child is sliding out of control with velocity v_c across a frozen lake. He runs head-on into another child, initially at rest, with 3 times the mass of the first child, who holds on so that the two now slide together. What is the velocity of the couple after the collision?

    (A) $2v_c$
    (B) v_c
    (C) $v_c/2$
    (D) $v_c/3$
    (E) $v_c/4$

7. An ice skater can rotate about a vertical axis with an angular velocity ω_0 by holding her arms straight out. She can then pull in her arms close to her body so that her angular velocity changes to $2\omega_0$, without the application of any external torque. What is the ratio of her final rotational kinetic energy to her initial rotational kinetic energy?

    (A) $\sqrt{2}$
    (B) 2
    (C) $2\sqrt{2}$
    (D) 4
    (E) 8

8. When a block of wood with a weight of 30 N is completely submerged under water the buoyant force on the block of wood from the water is 50 N. When the block is released it floats at the surface. What fraction of the block will then be visible above the surface of the water when the block is floating?

    (A) 1/15
    (B) 1/5
    (C) 1/3
    (D) 2/5
    (E) 3/5

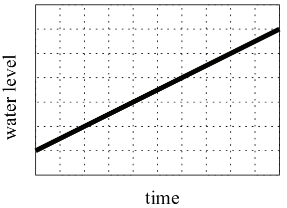
9. A spring has an equilibrium length of 2.0 meters and a spring constant of 10 newtons/meter. Alice is pulling on one end of the spring with a force of 3.0 newtons. Bob is pulling on the opposite end of the spring with a force of 3.0 newtons, in the opposite direction. What is the resulting length of the spring?

    (A) 1.7 m
    (B) 2.0 m
    (C) 2.3 m
    (D) 2.6 m
    (E) 8.0 m

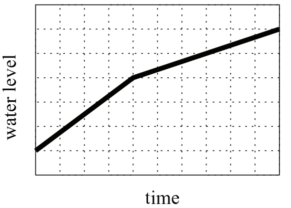
10. Which of the following changes will result in an *increase* in the period of a simple pendulum?

    (A) Decrease the length of the pendulum
    (B) Increase the mass of the pendulum
    (C) Increase the amplitude of the pendulum swing
    (D) Operate the pendulum in an elevator that is accelerating upward
    (E) Operate the pendulum in an elevator that is moving downward at constant speed.

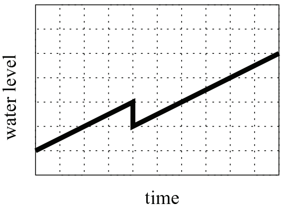
11. A large metal cylindrical cup floats in a rectangular tub half-filled with water. The tap is placed over the cup and turned on, releasing water at a constant rate. Eventually the cup sinks to the bottom and is completely submerged. Which of the following five graphs could represent the water level in the sink as a function of time?



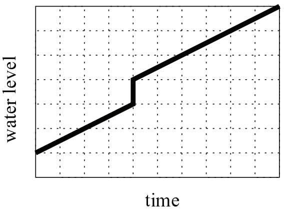
(A)



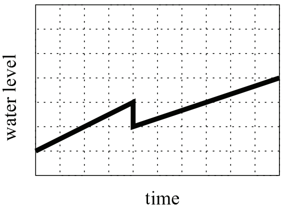
(B)



(C)

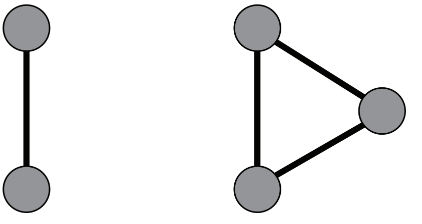


(D)



(E)

12. You are given a large collection of identical heavy balls and lightweight rods. When two balls are placed at the ends of one rod and interact through their mutual gravitational attraction (as is shown on the left), the compressive force in the rod is F. Next, three balls and three rods are placed at the vertexes and edges of an equilateral triangle (as is shown on the right). What is the compressive force in each rod in the latter case?



(A) $\frac{1}{\sqrt{3}}F$

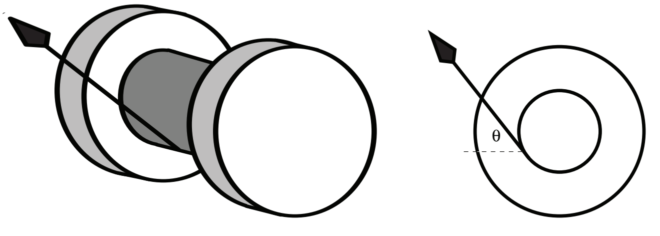
(B) $\frac{\sqrt{3}}{2}F$

(C) F

(D) $\sqrt{3}F$

(E) $2F$

13. The apparatus in the diagram consists of a solid cylinder of radius 1 cm attached at the center to two disks of radius 2 cm. It is placed on a surface where it can roll, but will not slip. A thread is wound around the central cylinder. When the thread is pulled at the angle $\theta = 90°$ to the horizontal (directly up), the apparatus rolls to the right. Which below is the largest value of θ for which it will not roll to the right when pulling on the thread?



(A) $\theta = 15°$

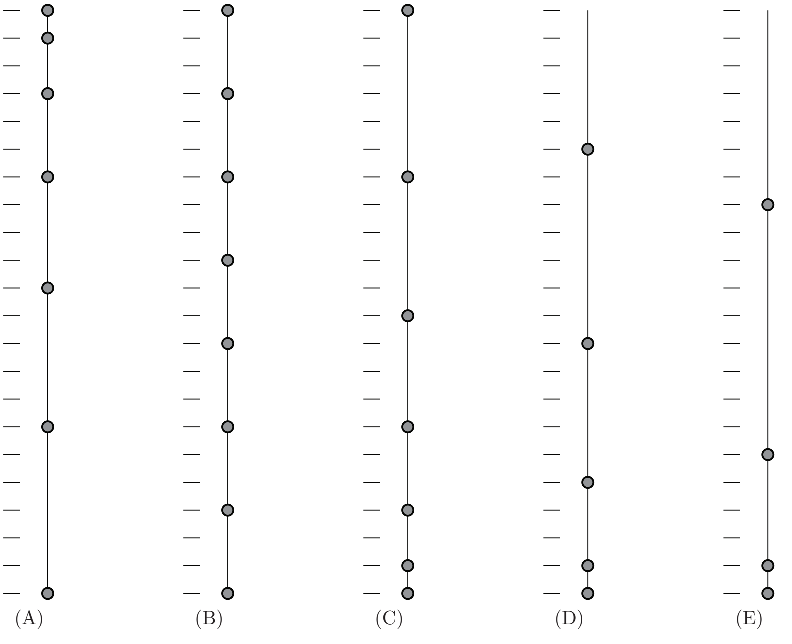
(B) $\theta = 30°$

(C) $\theta = 45°$

(D) $\theta = 60°$

(E) None, the apparatus will always roll to the right

14. You have 5 different strings with weights tied at various point, all hanging from the ceiling, and reaching down to the floor. The string is released at the top, allowing the weights to fall. Which one will create a regular, uniform beating sound as the weights hit the floor?

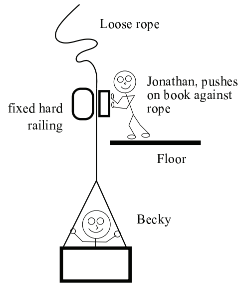


15. A vertical mass-spring oscillator is displaced 2.0 cm from equilibrium. The 100 g mass passes through the equilibrium point with a speed of 0.75 m/s. What is the spring constant of the spring?

    (A) 90 N/m
    (B) 100 N/m
    (C) 110 N/m
    (D) 140 N/m
    (E) 160 N/m

Questions 16 and 17 refer to the information and diagram below.

Jonathan is using a rope to lift a box with Becky in it; the box is hanging off the side of a bridge, Jonathan is on top. A rope is hooked up from the box and passes a fixed railing; Jonathan holds the box up by pressing the rope against the railing with a *massless*, *frictionless* physics textbook. The static friction coefficient between the rope and railing is μ_s; the kinetic friction coefficient between the rope and railing is $\mu_k < \mu_s$; the mass of the box and Becky combined is M; and the initial height of the bottom of the box above the ground is h. Assume a massless rope.



16. What magnitude force does Jonathan need to exert on the physics book to keep the rope from slipping?

    (A) Mg
    (B) $\mu_k Mg$
    (C) $\mu_k Mg/\mu_s$
    (D) $(\mu_s + \mu_k)Mg$
    (E) Mg/μ_s

17. Jonathan applies a normal force that is just enough to keep the rope from slipping. Becky makes a small jump, barely leaving contact with the floor of the box. Upon landing on the box, the force of the impact causes the rope to start slipping from Jonathan's hand. At what speed does the box smash into the ground? Assume Jonathan's normal force does not change.

    (A) $\sqrt{2gH}(\mu_k/\mu_s)$
    (B) $\sqrt{2gH}(1 - \mu_k/\mu_s)$
    (C) $\sqrt{2gH}\sqrt{\mu_k/\mu_s}$
    (D) $\sqrt{2gH}\sqrt{(1 - \mu_k/\mu_s)}$
    (E) $\sqrt{2gH}(\mu_s - \mu_k)$

18. A block of mass $m = 3.0$ kg slides down one ramp, and then up a second ramp. The coefficient of kinetic friction between the block and each ramp is $\mu_k = 0.40$. The block begins at a height $h_1 = 1.0$ m above the horizontal. Both ramps are at a 30° incline above the horizontal. To what height above the horizontal does the block rise on the second ramp?

    (A) 0.18 m
    (B) 0.52 m
    (C) 0.59 m
    (D) 0.69 m
    (E) 0.71 m

Questions 19 and 20 refer to the following information

A particle of mass 2.00 kg moves under a force given by

$$\vec{\mathbf{F}} = -(8.00\ \text{N/m})(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors in the x and y directions. The particle is placed at the origin with an initial velocity $\vec{\mathbf{v}} = (3.00\ \text{m/s})\hat{\mathbf{i}} + (4.00\ \text{m/s})\hat{\mathbf{j}}$.

19. After how much time will the particle first return to the origin?

    (A) 0.785 s
    (B) 1.26 s
    (C) 1.57 s
    (D) 2.00 s
    (E) 3.14 s

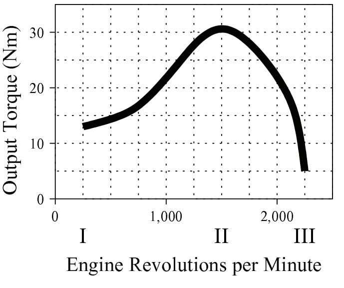
20. What is the maximum distance between the particle and the origin?

    (A) 2.00 m
    (B) 2.50 m
    (C) 3.50 m
    (D) 5.00 m
    (E) 7.00 m

21. An engineer is given a fixed volume V_m of metal with which to construct a spherical pressure vessel. Interestingly, assuming the vessel has thin walls and is always pressurized to near its bursting point, the amount of gas the vessel can contain, n (measured in moles), does not depend on the radius r of the vessel; instead it depends only on V_m (measured in m^3), the temperature T (measured in K), the ideal gas constant R (measured in J/(K$\cdot$mol)), and the tensile strength of the metal σ (measured in N/m^2). Which of the following gives n in terms of these parameters?

    (A) $n = \frac{2}{3}\frac{V_m\sigma}{RT}$
    (B) $n = \frac{2}{3}\frac{\sqrt[3]{V_m\sigma}}{RT}$
    (C) $n = \frac{2}{3}\frac{\sqrt[3]{V_m\sigma^2}}{RT}$
    (D) $n = \frac{2}{3}\frac{\sqrt[3]{V_m{}^2\sigma}}{RT}$
    (E) $n = \frac{2}{3}\sqrt[3]{\frac{V_m\sigma^2}{RT}}$

22. This graph depicts the torque output of a hypothetical gasoline engine as a function of rotation frequency. The engine is incapable of running outside of the graphed range.



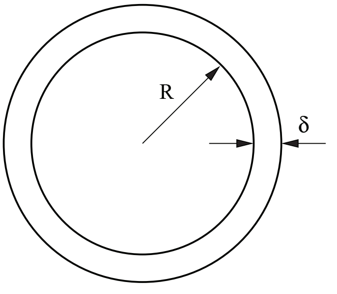
At what engine RPM (revolutions per minute) does the engine produce maximum power?

(A) I
(B) At some point between I and II
(C) II
(D) At some point between II and III
(E) III

23. A particle is launched from the surface of a uniform, stationary spherical planet at an angle to the vertical. The particle travels in the absence of air resistance and eventually falls back onto the planet. Spaceman Fred describes the path of the particle as a parabola using the laws of projectile motion. Spacewoman Kate recalls from Kepler's laws that every bound orbit around a point mass is an ellipse (or circle), and that the gravitation due to a uniform sphere is identical to that of a point mass. Which of the following best explains the discrepancy?

(A) Because the experiment takes place very close to the surface of the sphere, it is no longer valid to replace the sphere with a point mass.
(B) Because the particle strikes the ground, it is not in orbit of the planet and therefore can follow a non-elliptical path.
(C) Kate disregarded the fact that motions around a point mass may also be parabolas or hyperbolas.
(D) Kepler's laws only hold in the limit of large orbits.
(E) The path is an ellipse, but is very close to a parabola due to the short length of the flight relative to the distance from the center of the planet.

24. A turntable is supported on a Teflon ring of inner radius R and outer radius $R+\delta$ ($\delta \ll R$), as shown in the diagram. To rotate the turntable at a constant rate, power must be supplied to overcome friction. The manufacturer of the turntable wishes to reduce the power required without changing the rotation rate, the weight of the turntable, or the coefficient of friction of the Teflon surface. Engineers propose two solutions: increasing the width of the bearing (increasing δ), or increasing the radius (increasing R). What are the effects of these proposed changes?



(A) Increasing δ has no significant effect on the required power; increasing R increases the required power.
(B) Increasing δ has no significant effect on the required power; increasing R decreases the required power.
(C) Increasing δ increases the required power; increasing R has no significant effect on the required power.
(D) Increasing δ decreases the required power; increasing R has no significant effect on the required power.
(E) Neither change has a significant effect on the required power.

25. A hollow cylinder with a very thin wall (like a toilet paper tube) and a block are placed at rest at the top of a plane with inclination θ above the horizontal. The cylinder rolls down the plane without slipping and the block slides down the plane; it is found that both objects reach the bottom of the plane simultaneously. What is the coefficient of kinetic friction between the block and the plane?

(A) 0
(B) $\frac{1}{3}\tan\theta$
(C) $\frac{1}{2}\tan\theta$
(D) $\frac{2}{3}\tan\theta$
(E) $\tan\theta$

Answers, Problem Difficulty, and Topics

ToC

2011.1	A	⋆	linear motion
2011.2	E	⋆	linear motion
2011.3	D	⋆	linear motion
2011.4	B	⋆	linear motion
2011.5	B	⋆	circular motion
2011.6	E	⋆	collisions
2011.7	B	⋆	conservation of angular momentum
2011.8	D	⋆	Archimedes' Principle
2011.9	C	⋆	springs, Hooke's Law
2011.10	C	⋆ ⋆ ⋆	simple pendulum
2011.11	C	⋆ ⋆	Archimedes' Principle
2011.12	C	⋆	gravitation
2011.13	D	⋆ ⋆	torque, static friction, tension
2011.14	D	⋆ ⋆	linear motion
2011.15	D	⋆ ⋆ ⋆	springs, preloading, energy
2011.16	E	⋆	static friction
2011.17	D	⋆ ⋆ ⋆	kinetic friction, work, conservation of energy
2011.18	A	⋆	kinetic friction, work, conservation of energy
2011.19	C	⋆ ⋆	springs, oscillations
2011.20	B	⋆ ⋆	springs, conservation of energy
2011.21	A	⋆	dimensional analysis
2011.22	D	⋆ ⋆	power, torque
2011.23	E	⋆ ⋆ ⋆	Kepler's Laws
2011.24	A	⋆ ⋆	power, kinetic friction
2011.25	C	⋆ ⋆ ⋆	rolling motion, kinetic friction

Solutions

ToC

2011.1. This problem can be solved using arithmetic, i.e., just numbers, without introducing variables. However, working with variables, using algebra until the final formula, and only then plugging the numbers in, has many advantages. For that reason, let us denote the average speed by $\bar{v}$, the constant driving speed by v, the break duration by T, the unknown total driving time by t, and the unknown distance by s. Then,

$$\bar{v} = \frac{s}{t} \qquad \Rightarrow \qquad s = \bar{v}t.$$

Also,

$$s = v\,(t - T).$$

We now have two equations in two unknowns, s and t. Since we do not need t in this problem, we want to eliminate it by writing the expression for it using the simpler of the two equations – in this case from the first equation[1],

$$t = \frac{s}{\bar{v}}.$$

When we plug this into the second equation, we get

$$s = v\left(\frac{s}{\bar{v}} - T\right).$$

Thus, we turned our system of two equations in two unknowns into a single equation in one unknown. We rearrange the last equation in order to collect all terms containing s on the same side of the equation,

$$s\,\frac{v}{\bar{v}} - s = vT \quad \Rightarrow \quad s\left(\frac{v}{\bar{v}} - 1\right) = vT \quad \Rightarrow \quad s = \frac{v\bar{v}T}{v - \bar{v}}.$$

Plugging in the numbers (do not forget to first convert 20 min into 1/3 h), we get

$$s = \frac{22\text{ km/h} \cdot 17.5\text{ km/h} \cdot 1/3\text{ h}}{22\text{ km/h} - 17.5\text{ km/h}} = 28.5\text{ km},$$

so the correct answer is A.

[1] There is nothing wrong with eliminating s first and solving for t, then going back and solving for s, but that approach has an extra, unnecessary step, because we do not really need the value for t in this problem.

2011.2. All three graphs are velocity vs. time graphs, so the magnitude of the average acceleration is the absolute value of the average slope of the graph. In general, average acceleration is $a = \frac{v_f - v_i}{\Delta t}$. For our graphs:

$$a_I = \frac{-2 \text{ m/s} - 0 \text{ m/s}}{10 \text{ s}} = -0.2 \text{ m/s}^2, \quad a_{II} = \frac{0 \text{ m/s} - 2 \text{ m/s}}{10 \text{ s}} = -0.2 \text{ m/s}^2,$$

$$a_{III} = \frac{2 \text{ m/s} - 0 \text{ m/s}}{10 \text{ s}} = 0.2 \text{ m/s}^2.$$

We see that $|a_I| = |a_{II}| = |a_{III}| = 0.2 \text{ m/s}^2$, so the answer is E.

2011.3. Directly from the graphs, we see that v_{max} in graph I is 4 m/s, while for both graphs II and III we have 2 m/s. Therefore, the answer is D.

2011.4. In a velocity vs. time graph, the total *distance traveled* is the total area between the graphs and the time axis, where the parts above and under the time axis both count as positive[2]. For the three graphs they are[3]:

- $s_I = \frac{1}{2} \cdot 4 \text{ m/s} \cdot 4 \text{ s} + \frac{1}{2} \cdot 4 \text{ m/s} \cdot 4 \text{ s} + \frac{1}{2} \cdot 2 \text{ m/s} \cdot 2 \text{ s} = 18 \text{ m/s}$,
- $s_{II} = 2 \text{ m/s} \cdot 9 \text{ s} + \frac{1}{2} \cdot 2 \text{ m/s} \cdot 1 \text{ s} = 19 \text{ m/s}$,
- $s_{III} = \frac{1}{2} \cdot 2 \text{ m/s} \cdot 6 \text{ s} + 2 \text{ m/s} \cdot 4 \text{ s} = 14 \text{ m/s}$.

We see that $s_{II} > s_I > s_{III}$, so the answer is B.

2011.5. In this approximation, the Earth's *speed* is constant, so you may be tricked into thinking that its acceleration is 0. It is not, however, because its *velocity* is not constant[4]: its magnitude is constant (speed), but its direction keeps changing. It can be shown that the acceleration needed to keep an object on a circular trajectory is always directed towards the center of the circle (that is why we call it radial, or central, or centripetal acceleration) and its magnitude is $a_c = \frac{v^2}{r}$. We can determine v from the circumference of the Earth's orbit and its orbital period, i.e., the duration of Earth's year: $v = 2\pi r/T = 2\pi \cdot 150 \cdot 10^6 \text{ km} / 365.25 \text{ days} \approx 30 \text{ km/s}$ (yes, that's roughly how fast we are circling the Sun!), so $a_c = 0.006 \text{ m/s}^2$, and the answer is B.

Note: Compared to the gravitational acceleration we experience on the surface of the Earth due to Earth's gravity[5], the result of this problem is more than 1000 times smaller and is negligible for most practical purposes.

[2] Note that for the total *displacement*, areas under the time axis count as negative.

[3] Here we partition the total areas into triangles and rectangles. Alternatively, we can treat these areas as trapezoids.

[4] *Velocity* is a vector, while *speed* is a scalar (the magnitude of the velocity vector).

[5] We denote the magnitude of the gravitational acceleration on the surface of the Earth by g. Its value is around 9.8 m/s^2 and varies with altitude, latitude, and other factors. In the $F = ma$ competition we are asked to use the approximation of 10 m/s^2, probably to simplify the calculations.

2011.6. The two children slide together after the collision, so this is a perfectly inelastic collision, and we can use the conservation of linear momentum. If their velocities before the collision were $v_1 = v_c$ and $v_2 = 0$, while their masses were $m_1 = m$ and $m_2 = 3m$, we can write

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) w,$$

where w is their velocity after the collision. From this,

$$w = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{m v_c}{4m} = \frac{v_c}{4},$$

so the answer is E.

2011.7. The conservation law that applies here is the conservation of angular momentum,

$$L = I\omega = \text{const.}$$

Since we have $\omega_2 = 2\omega_1$, we must also have $I_2 = \frac{1}{2} I_1$. Therefore,

$$\frac{E_{k2}}{E_{k1}} = \frac{\frac{1}{2} I_2 \omega_2^2}{\frac{1}{2} I_1 \omega_1^2} = 2 \qquad \Rightarrow \qquad \text{the answer is B.}$$

Note: Where does this increase in kinetic energy come from? If you ever tried to spin on ice or at least on a rotating chair, you will have noticed that it takes some effort to pull your arms in, so the skater is doing the work that gets converted into the additional rotational kinetic energy.

2011.8. Archimedes' Principle states that the buoyancy on an object is equal to the weight of the water it displaces. In this problem, we see that the initial buoyancy B is greater than the object's weight W, so there must have been an additional force

$$F = B - W = 50 \text{ N} - 30 \text{ N} = 20 \text{ N}$$

that kept the object completely submerged. Once that force was removed, the object started floating. The part of the object that is now out of the water, previously displaced the weight of $F = 20$ N of water. The part of the object that is still in water, displaces the weight of 30 N of water to match the object's weight W. Therefore, the fraction of the object that is out of water is

$$\frac{F}{W+F} = \frac{B-W}{B} = 1 - \frac{W}{B} = 1 - \frac{3}{5} = \frac{2}{5} \qquad \Rightarrow \qquad \text{the answer is D.}$$

2011.9. This problem may be confusing because we usually consider a spring attached on one side to a wall and pulled on the other side by an experimenter. But nothing is different if we have two experimenters pulling on opposite sides of the spring with equal forces, because when you think about it, the wall pulls on the spring with the force that matches the single experimenter's force. Therefore, the new length is

$$x + \Delta x = x + F/k = 2.3 \text{ m} \quad \Rightarrow \quad \text{the result is C.}$$

Note 1: If you are not convinced that a static wall and the second experimenter play the same role, consider the situation with two experimenters again. Because of symmetry, the center of the spring does not move, so we can imagine cutting the spring in two halves. Then we have two springs, each as if being fixed to a wall on one end. Of course, the extension we get with one spring half here will be doubled for the final result. Also, remember that each of the spring halves will have a spring constant $k' = 2k$. Therefore, the total extension will be $\Delta x = 2 \cdot F/k' = F/k$, just as in the original analysis.

Note 2: Think about what happens if we again have two experimenters, but this time they try to pull on the opposite ends of the spring with unequal forces $F_1 \neq F_2$.

2011.10. First, a few facts about a simple pendulum:

- For "small amplitude" oscillations[6], the period of a simple pendulum is well approximated by

$$T = 2\pi\sqrt{\frac{\ell}{g}},$$

  and does not depend on the mass of the bob nor on the amplitude, only on the length of the rod.

- The pendulum period increases slightly as θ increases, because the approximation $\sin\theta \approx \theta$, where θ is in radians, used in derivation of the above formula for T, becomes less accurate for greater values of θ. Thus, the magnitudes of the force and acceleration become less than what they would be for a harmonic oscillator.

- The formula for T that includes the dependence on the amplitude θ_0 (here expressed in radians) is

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \dots\right) \quad \Rightarrow \quad T \text{ increases with } \theta_0.$$

[6] For most practical purposes, we can say that "small amplitude" means less than around 20^o for the maximum angle. This is because the true value of the pendulum period will deviate from the formula by only 1% when the amplitude is 22.3^o. If the amplitude is only 7.25^o, the deviation from the formula is only $1/1000 = 0.1\%$.

- Vertical acceleration a is added to g if we are accelerating up, and subtracted from g if we are accelerating down. This is consistent with the extreme case of a free fall ($a = g$ going down), when the pendulum stops swinging and the period becomes infinitely long.

With all this, we see for the answer choices:

A is incorrect, because $T \propto \sqrt{\ell}$.

B is incorrect, because T does not depend on the mass.

C is correct, as discussed above.

D is incorrect, as upward acceleration effectively increases g and $T \propto \sqrt{1/g}$.

E is incorrect, because with constant speed, there is no acceleration.

We see that the correct answer is C.

2011.11. As the cup is being filled with water, initially it keeps floating, but it gets deeper and deeper, thus raising the level of water in the tub at a constant rate. When the edge of the cup reaches the level of tub water, some water from the tub will quickly fill it up, thus causing a sudden drop in the tub water level. After that, the level of the tub water will continue to rise at a constant rate. With this, we reduced the choices to C and E.

The difference between C and E is about whether the rates of water level increase are the same before and after the cup sinks. The key observation here is that the cup and the tub are cylindrical and rectangular, respectively, so their horizontal cross-sections are constant. As the cup is being filled, the difference Δh between the interior and exterior water levels remains constant[7]. If the volume of water inside the cup changes by ΔV over time Δt, the level inside the cup (as measured from the bottom of the cup) increases by $\Delta \ell = \Delta V/a$, where a is the area of the cup's bottom. For Δh to remain constant, the cup goes down relative to the external water level by the same amount $\Delta \ell$. This displaces $a\Delta \ell = \Delta V$ of water, which raises the water level in the tub by $\Delta L = \Delta V/A$, where A is the area of the tub's bottom[8]. This is the same

[7]Let the bottom of the cup be distance H lower than the external water level. Let the mass of the cup be m_c and the mass of water inside the cup m_i. Let the area of the cup's bottom be a and let the water level inside the cup be Δh below the external water level. Let us also neglect the weight of the air in the cup under the external water level. Then the two weights are balanced by the buoyancy (ρ is the density of water and V is the volume of water displaced by the cup): $m_c g + m_i g = \rho g V \;\Rightarrow\; m_c = \rho V - m_i = \rho a H - \rho a(H - \Delta h) = \rho a \Delta h \;\Rightarrow\; \Delta h = m_c/(\rho a)$. Therefore, Δh does not depend on how much water is in the cup.

[8]Note that $\Delta L \neq \Delta V/(A - a)$, because Δh is constant.

increase in the external water level over time Δt as when ΔV goes directly into the tub. Therefore, the answer is C.

Note: Over time Δt, the cup goes deeper (with respect to the external water level) by $\Delta \ell = \Delta V/a$. At the same time, the external water level goes up by $\Delta L = \Delta V/A$, so the net motion of the cup over time Δt is down by amount

$$\Delta y = \Delta \ell - \Delta L = \Delta V(1/a - 1/A).$$

2011.12. If we name the three balls A, B, and C, the magnitude of the resultant force on A due to gravity from B and C is (based on the rule of parallelogram) $F_{BC} = 2F\sqrt{3}/2 = F\sqrt{3}$. To get the individual compressive forces in rods AB and AC, we have to decompose $\vec{F}_{BC}$ back in the same two directions as its constituent forces. Therefore, the compressive forces in the triangular configuration are equal to the compressive forces in the case of only two balls on rods, and the answer is C.

Note: Alternatively, to explain why the forces are all F, we notice that there is one rod for each gravitationally attracting pair, so each rod counteracting the respective F due to gravity leads us to C.

2011.13. The only two forces that influence motion in this problem are the string tension T and static friction F_s. Let us find the angle θ for which the forces and torques are balanced. Horizontally, we see that the static friction must be balanced by the horizontal component of the string tension T, so $F_s = T\cos\theta$. For torques to be balanced, we must have (with R denoting the outer radius and r the inner radius) $Tr = F_s R$. Combining these two equations we get $\cos\theta = \frac{r}{R} = \frac{1}{2} \Rightarrow \theta = 60^o$, so the answer is D.

2011.14. For the weights to hit the ground in a steady rhythm, the higher we look, the greater the distance between the balls must be, otherwise the beat rate will be speeding up. Therefore, answer choices A and B are out. To figure out which one of the three remaining choices is correct, recall Galileo's discovery that under constant acceleration, for example with balls sliding on an incline, as in this video at the *Virtual Museo Galileo* website:

`https://catalogue.museogalileo.it/multimedia/InclinedPlane.html`,

The distances covered in regular time intervals differ by odd multiples of the distance covered in the first time interval[9]. The distances between balls in D are exactly what "Galileo's trick" prescribes: $1, 3, 5, 7, \ldots$, so the answer is D.

[9] Galileo knew that the sum of the first n odd integers is n^2, i.e., $1+3+5+\ldots+2n-1 = n^2$, which helped him discover that in uniformly accelerated motion starting from rest, the total distance traveled is proportional to the square of the time. That led Galileo to the formula $s = \dfrac{at^2}{2}$.

2011.15. This is a tricky problem because of spring preloading. We will show that in the end gravity does not matter in this problem.

Approach 1: For a quick solution, first consider what happens if there was no gravity. Without gravity, the equilibrium point would be higher, while with gravity, the spring is extended a bit, so that the spring force matches the weight of the mass. With this extension due to gravity, the mass is in a weightless state and no work is done by or against gravity in this problem[10]. With this, we write (with x denoting the displacement from the equilibrium with gravity):

$$\frac{mv^2}{2} = \frac{kx^2}{2} \quad \Rightarrow \quad k = \frac{mv^2}{x^2} = 140\ \mathrm{N/m} \quad \Rightarrow \quad \text{so the answer is D.}$$

Approach 2: The following approach is more systematic and justifies the above quick thinking. The energy stored in the spring when it is in equilibrium with gravity is

$$E_0 = \frac{ky^2}{2},$$

where y is the extension of the spring due to gravity (the difference between the spring equilibria with and without gravity), so

$$ky = mg.$$

When the spring is extended or compressed by displacement x from the equilibrium with gravity, the energy stored in the spring is

$$E_1 = \frac{k(y+x)^2}{2}.$$

Once the mass is released and goes through the equilibrium with gravity, it will have speed v, so E_1 gets transformed into gravitational potential energy, kinetic energy, and the remaining spring energy:

$$E_1 = \frac{k(y+x)^2}{2} = mgx + \frac{mv^2}{2} + \frac{ky^2}{2}.$$

Multiplying out the left-hand side leads us to (note that $ky = mg$)

$$\cancel{\frac{ky^2}{2}} + \cancel{kyx} + \frac{kx^2}{2} = \cancel{mgx} + \frac{mv^2}{2} + \cancel{\frac{ky^2}{2}},$$

so we find (just as in Approach 1)

$$\frac{mv^2}{2} = \frac{kx^2}{2} \quad \Rightarrow \quad k = \frac{mv^2}{x^2} = 140\ \mathrm{N/m} \quad \Rightarrow \quad \text{so the answer is D.}$$

[10]Linearity of $F = -kx$ plays a big role in this conclusion. Gravity is adding a constant, which does not change the slope of the line, leading to the potential energy formula $kx^2/2$.

2011.16. The force that is keeping the box from moving down is the static friction between the rope and the railing. If Jonathan is pressing the book onto the rope with force F, the static friction is $F_s = \mu_s F$. For it to match the weight of the box with Becky in it, we must have

$$Mg = F_s = \mu_s F \quad \Rightarrow \quad F = \frac{Mg}{\mu_s} \quad \Rightarrow \quad \text{the answer is E.}$$

2011.17. After Becky jumps, the rope begins to move and the friction is now kinetic. As the box and Becky in it move down, they are pulled down by gravity and slowed down by kinetic friction. The gravitational potential energy is converted into kinetic energy and the work against kinetic friction, so we can write

$$\begin{aligned} MgH &= \frac{Mv^2}{2} + F_k H \\ &= \frac{Mv^2}{2} + \mu_k F H \\ &= \frac{Mv^2}{2} + \mu_k \frac{Mg}{\mu_s} H. \end{aligned}$$

Multiplying both sides by $2/M$ and rearranging, we get

$$v^2 = 2gH\left(1 - \frac{\mu_k}{\mu_s}\right) \qquad \Rightarrow \qquad \text{the answer is D.}$$

2011.18. The initial potential energy of the body is converted into the final potential energy and into work against kinetic friction $F_k = \mu_k mg\cos\theta$. Denoting by s_1 and s_2 the displacements along the slope, we have

$$mgh_1 = mgh_2 + (\mu_k mg\cos\theta)s_1 + (\mu_k mg\cos\theta)s_2.$$

Since $s_1 = h_1/\sin\theta$ and $s_2 = h_2/\sin\theta$, we can write

$$mgh_1 = mgh_2 + \frac{\mu_k mg\cos\theta}{\sin\theta}(h_1 + h_2).$$

From this we find

$$h_2 = h_1\,\frac{\sin\theta - \mu_k\cos\theta}{\sin\theta + \mu_k\cos\theta} = 0.18\ \text{m} \qquad \Rightarrow \qquad \text{the answer is A.}$$

2011.19. This problem describes a 2D motion under the given force field $\vec{F}$. Looking at the x and y components of $\vec{F}$, we see that they are equivalent to springs with equal spring constants, $k_x = k_y = 8$ N/m. Therefore, this "2D spring-mass system" will have oscillations with period

$$T = 2\pi\sqrt{\frac{m}{k}} = 3.14\ \text{s}.$$

The question asks for the time when the mass will first return to the origin (equilibrium point) of the system. This will happen after half of the period, $t = T/2 = 1.57$ s, so the answer is C.

Note: What would the answer be if the two spring constants were not equal? Compare the following two cases:

(a) $k_x = 8$ N/m and $k_y = 2$ N/m, vs.

(b) $k_x = 8$ N/m and $k_y = 8\pi^2$ N/m (this case is of purely theoretical interest).

2011.20. For the two components of this "2D spring-mass system" we can write

$$\frac{mv_{0x}^2}{2} = \frac{kx_{max}^2}{2} \qquad \text{and} \qquad \frac{mv_{0y}^2}{2} = \frac{ky_{max}^2}{2},$$

therefore,

$$x_{max} = v_{0x}\sqrt{\frac{m}{k}} = \frac{3}{2}\text{ m} \qquad \text{and} \qquad y_{max} = v_{0y}\sqrt{\frac{m}{k}} = 2\text{ m}.$$

The maximum distance from the origin is then

$$r_{max} = \sqrt{x_{max}^2 + y_{max}^2} = \sqrt{\frac{3^2+4^2}{2^2}}\text{ m} = \frac{5}{2}\text{ m} = 2.5\text{ m} \quad \Rightarrow \quad \text{the answer is B.}$$

2011.21. This is one of those problems that looks very scary at first sight, but is actually fairly easy once we realize it is about dimensional (physical units) analysis. The unit for n is $[n] =$ mol, so we expect that the right-hand side of the correct formula will also have the same unit. The formula in A yields (note that $\text{J} = \text{N} \cdot \text{m}$)

$$\left[\frac{2}{3}\frac{V_m\sigma}{RT}\right] = \frac{\text{m}^3 \cdot \text{N/m}^2}{\text{J/(K mol)} \cdot \text{K}} = \text{mol.}$$

All other formulas fail to produce the correct unit, so the answer is A.

2011.22. Analogously to the formula $P = Fv$ used in translatory motion, here we use the formula

$$P = \tau\omega.$$

According to the graph, this product will be at its maximum power a bit to the right from the peak at II (because the points with equal power lie on hyperbolas $\tau = P/\omega$). Therefore, the answer is D.

2011.23. Spacewoman Kate is completely right, the trajectory here is indeed an ellipse, and yes, in terms of gravity, this planet (and any other spherically symmetric body) can be replaced by all of its mass concentrated at its center. If the length of the flight is short with respect to the radius of the planet, than the lines of force in the planet's gravitational field can be assumed to be parallel and have constant magnitude, allowing the relevant part of the ellipse to be well approximated with a parabola, just like what we do in the study of projectile motion. The correct answer choice is E.

Note: Let us comment on the answer choice C. While it is true that Kepler's First Law allows parabolas and hyperbolas, the true trajectory in this case must be an ellipse, because we are told that the particle will fall back to the planet, so it has a bound trajectory. Had we launched the particle with velocity equal or greater to the planet's *escape velocity* (also known as the *second cosmic velocity*),

$$v_2 = \sqrt{2gR} = \sqrt{\frac{2GM}{R}},$$

where g is the gravitational acceleration on the surface of the planet, R is its radius, G is the universal gravitational constant, and M is the mass of the planet, the particle's trajectory would be a parabola or a hyperbola, respectively, and the particle would have enough kinetic energy to leave the planet and not come back.

2011.24. The idea here is that the Teflon ring is under the turntable and supports it as it rotates, but it also acts on the turntable with friction. The problem asks for the power needed to overcome friction, i.e., to spin the turntable at a constant angular velocity ω. This can be computed as

$$P = F_k v = \mu W R \omega.$$

We are given that μ, W, and ω are to remain constant. Since $\delta \ll R$, changing it does not matter. However, R is proportional to the required power. Therefore, the answer is A.

2011.25. The acceleration of the block is given by

$$a_1 = g \sin\theta - \mu_k g \cos\theta,$$

reflecting the fact that it is pulled down the slope by the component of the block's weight parallel to the slope, and that it is slowed down by kinetic friction, which is proportional to the component of the block's weight normal (perpendicular) to the slope.

The acceleration of the rolling cylinder will depend on its moment of inertia. In general, if an object that is rolling without slipping has moment of inertia $I = cmr^2$, where m is its mass, r its radius, and c is a constant depending on the object's shape (for a hollow cylinder $c = 1$), its acceleration due to gravity on an incline will be[11]

$$a_2 = \frac{g\sin\theta}{1+c}.$$

Equating a_1 and a_2 (we use $c = 1$), we get

$$g\sin\theta - \mu_k g\cos\theta = \frac{g\sin\theta}{2}$$

$$\sin\theta = 2\mu_k\cos\theta$$

$$\mu_k = \frac{1}{2}\tan\theta,$$

so the correct answer is C.

[11] We derive this fact in Problem 2012.10.

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Year 2012

$\boldsymbol{F = ma}$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2012

2012 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers will result in a deduction of $\frac{1}{4}$ point. There is no penalty for leaving an answer blank.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2012.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Contributors to this years exam include Jiajia Dong, Qiuzi Li, Paul Stanley, Warren Turner, former US Team members Marianna Mao, Andrew Lin, Steve Byrnes, Adam Jermyn, Ante Qu, Alok Saxena, Tucker Chan, Kenan Diab, Jason LaRue.

1. Consider a dripping faucet, where the faucet is 10 cm above the sink. The time between drops is such that when one drop hits the sink, one is in the air and another is about to drop. At what height above the sink will the drop in the air be right as a drop hits the sink?

   (A) Between 0 and 2 cm,
   (B) Between 2 and 4 cm,
   (C) Between 4 and 6 cm,
   (D) Between 6 and 8 cm,
   (E) Between 8 and 10 cm,

2. A cannonball is launched with initial velocity of magnitude v_0 over a horizontal surface. At what minimum angle θ_{min} above the horizontal should the cannonball be launched so that it rises to a height H which is larger than the horizontal distance R that it will travel when it returns to the ground?

   (A) $\theta_{\text{min}} = 76^\circ$
   (B) $\theta_{\text{min}} = 72^\circ$
   (C) $\theta_{\text{min}} = 60^\circ$
   (D) $\theta_{\text{min}} = 45^\circ$
   (E) There is no such angle, as $R > H$ for all range problems.

3. An equilateral triangle is sitting on an inclined plane. Friction is too high for it to slide under any circumstance, but if the plane is sloped enough it can "topple" down the hill. What angle incline is necessary for it to start toppling?

   (A) 30 degrees
   (B) 45 degrees
   (C) 60 degrees
   (D) It will topple at any angle more than zero
   (E) It can never topple if it cannot slide

4. A particle at rest explodes into three particles of equal mass in the absence of external forces. Two particles emerge at a right angle to each other with equal speed v. What is the speed of the third particle?

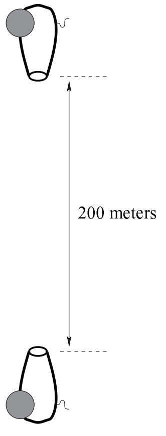
   (A) v
   (B) $\sqrt{2}v$
   (C) $2v$
   (D) $2\sqrt{2}v$
   (E) The third particle can have a range of different speeds.

5. A 12 kg block moving east at 4 m/s collides head on with a 6 kg block that is moving west at 2 m/s. The two blocks move together after the collision. What is the *loss* in kinetic energy in this collision?

   (A) 36 J
   (B) 48 J
   (C) 60 J
   (D) 72 J
   (E) 96 J

The following information applies to questions 6 and 7

Two cannons are arranged vertically, with the lower cannon pointing upward (towards the upper cannon) and the upper cannon pointing downward (towards the lower cannon), 200m above the lower cannon. Simultaneously, they both fire. The muzzle velocity of the lower cannon is 25m/s and the muzzle velocity of the upper cannon is 55m/s.



6. How long after the cannons fire do the projectiles collide?

   (A) 2.2 s
   (B) 2.5 s
   (C) 3.6 s
   (D) 6.7 s
   (E) 8.0 s

7. How far beneath the top cannon do the projectiles collide?

   (A) 31 m
   (B) 67 m
   (C) 110 m
   (D) 140 m
   (E) 170 m

8. A block of mass $m = 3.0$ kg is moving on a horizontal surface towards a massless spring with spring constant $k = 80.0$ N/m. The coefficient of kinetic friction between the block and the surface is $\mu_k = 0.50$. The block has a speed of 2.0 m/s when it first comes in contact with the spring. How far will the spring be compressed?

   (A) 0.19 m
   (B) 0.24 m
   (C) 0.39 m
   (D) 0.40 m
   (E) 0.61 m

9. A uniform spherical planet has radius R and the acceleration due to gravity at its surface is g. What is the escape velocity of a particle from the planet's surface?

   (A) $\frac{1}{2}\sqrt{gR}$
   (B) $\sqrt{gR}$
   (C) $\sqrt{2gR}$
   (D) $2\sqrt{gR}$
   (E) The escape velocity cannot be expressed in terms of g and R alone.

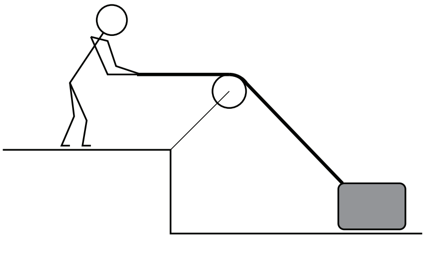
10. Four objects are placed at rest at the top of an inclined plane and allowed to roll without slipping to the bottom in the absence of rolling resistance and air resistance.

    - Object A is a solid brass ball of diameter d.
    - Object B is a solid brass ball of diameter $2d$.
    - Object C is a hollow brass sphere of diameter d.
    - Object D is a solid aluminum ball of diameter d. (Aluminum is less dense than brass.)

    The balls are placed so that their centers of mass all travel the same distance. In each case, the time of motion T is measured. Which of the following statements is correct?

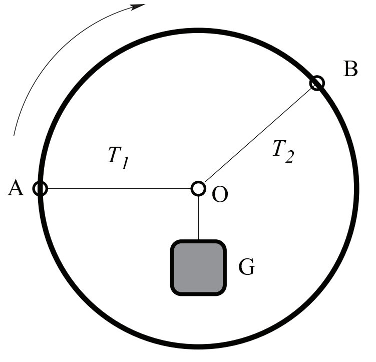
   (A) $T_B > T_C > T_A = T_D$
   (B) $T_A = T_B = T_C > T_D$
   (C) $T_B > T_A = T_C = T_D$
   (D) $T_C > T_A = T_B = T_D$
   (E) $T_A = T_B = T_C = T_D$

11. As shown below, Lily is using the rope through a fixed pulley to move a box with constant speed v. The kinetic friction coefficient between the box and the ground is $\mu < 1$; assume that the fixed pulley is massless and there is no friction between the rope and the fixed pulley. Then, while the box is moving, which of the following statements is correct?



   (A) The magnitude of the force on the rope is constant.
   (B) The magnitude of friction between the ground and the box is decreasing.
   (C) The magnitude of the normal force of the ground on the box is increasing.
   (D) The pressure of the box on the ground is increasing.
   (E) The pressure of the box on the ground is constant.

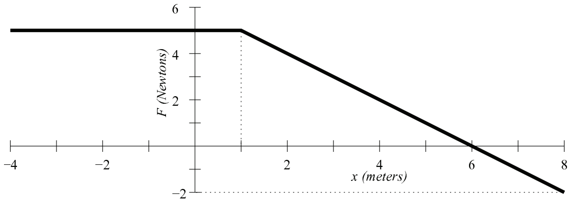
12. A rigid hoop can rotate about the center. Two massless strings are attached to the hoop, one at A, the other at B. These strings are tied together at the center of the hoop at O, and a weight G is suspended from that point. The strings have a fixed length, regardless of the tension, and the weight G is only supported by the strings. Originally OA is horizontal.



Now, the outer hoop will start to slowly rotate 90° clockwise until OA will become vertical, while keeping the angle between the strings constant and keeping the object static. Which of the following statements about the tensions T_1 and T_2 in the two strings is correct?

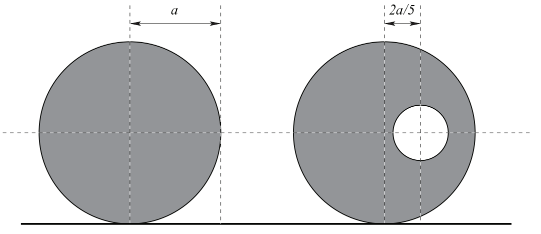
(A) T_1 always decreases.
(B) T_1 always increases.
(C) T_2 always increases.
(D) T_2 will become zero at the end of the rotation.
(E) T_2 first increases and then decreases.

13. Shown below is a graph of the x component of force versus position for a 4.0 kg cart constrained to move in one dimension on the x axis. At $x = 0$ the cart has a velocity of -3.0 m/s (in the negative direction). Which of the following is closest to the maximum speed of the cart?



(A) 1.6 m/s
(B) 2.5 m/s
(C) 3.0 m/s
(D) 4.0 m/s
(E) 4.2 m/s

14. A uniform cylinder of radius a originally has a weight of 80 N. After an off-axis cylinder hole at $2a/5$ was drilled through it, it weighs 65 N. The axes of the two cylinders are parallel and their centers are at the same height.



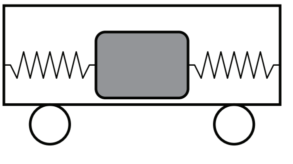
A force T is applied to the top of the cylinder horizontally. In order to keep the cylinder is at rest, the magnitude of the force is closest to:

(A) 6 N
(B) 10 N
(C) 15 N
(D) 30 N
(E) 38 N

15. A car of mass m has an engine that provides a constant power output P. Assuming no friction, what is the maximum constant speed $v_{\max}$ that this car can drive up a long incline that makes an angle θ with the horizontal?

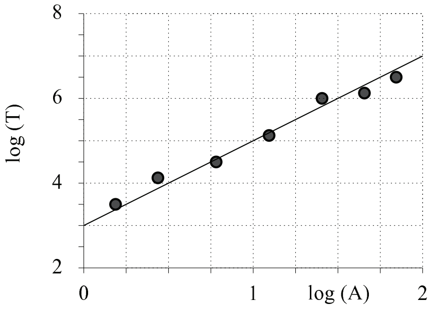
(A) $v_{\max} = P/(mg \sin\theta)$
(B) $v_{\max} = P^2 \sin\theta/mg$
(C) $v_{\max} = \sqrt{2P/mg}/\sin\theta$
(D) There is no maximum constant speed.
(E) The maximum constant speed depends on the length of the incline.

16. Inside a cart that is accelerating horizontally at acceleration $\vec{a}$, there is a block of mass M connected to two light springs of force constants k_1 and k_2. The block can move without friction horizontally. Find the vibration frequency of the block.



(A) $\frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{M} + a}$
(B) $\frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{(k_1+k_2)M}}$
(C) $\frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{(k_1+k_2)M} + a}$
(D) $\frac{1}{2\pi}\sqrt{\frac{|k_1-k_2|}{M}}$
(E) $\frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{M}}$

17. Shown below is a log/log plot for the data collected of amplitude and period of oscillation for certain non-linear oscillator.



According to the data, the relationship between period T and amplitude A is best given by

(A) $T = 1000A^2$
(B) $T = 100A^3$
(C) $T = 2A + 3$
(D) $T = 3\sqrt{A}$
(E) Period is independent of amplitude for oscillating systems

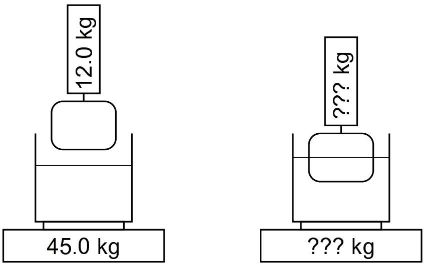
18. A mass hangs from the ceiling of a box by an ideal spring. With the box held fixed, the mass is given an initial velocity and oscillates with purely vertical motion. When the mass reaches the lowest point of its motion, the box is released and allowed to fall. To an observer inside the box, which of the following quantities **does not** change when the box is released? Ignore air resistance.

(A) The amplitude of the oscillation
(B) The period of the oscillation
(C) The maximum speed reached by the mass
(D) The height at which the mass reaches its maximum speed
(E) The maximum height reached by the mass

19. A 1,500 Watt motor is used to pump water a vertical height of 2.0 meters out of a flooded basement through a cylindrical pipe. The water is ejected though the end of the pipe at a speed of 2.5 m/s. Ignoring friction and assuming that all of the energy of the motor goes to the water, which of the following is the closest to the radius of the pipe? The density of water is $\rho = 1000$ kg/m^3.

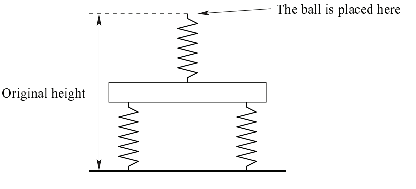
(A) 1/3 cm
(B) 1 cm
(C) 3 cm
(D) 10 cm
(E) 30 cm

20. A container of water is sitting on a scale. Originally, the scale reads $M_1 = 45$ kg. A block of wood is suspended from a second scale; originally the scale read $M_2 = 12$ kg. The density of wood is 0.60 g/cm^3; the density of the water is 1.00 g/cm^3. The block of wood is lowered into the water until half of the block is beneath the surface. What is the resulting reading on the scales?



(A) $M_1 = 45$ kg and $M_2 = 2$ kg.
(B) $M_1 = 45$ kg and $M_2 = 6$ kg.
(C) $M_1 = 45$ kg and $M_2 = 10$ kg.
(D) $M_1 = 55$ kg and $M_2 = 6$ kg.
(E) $M_1 = 55$ kg and $M_2 = 2$ kg.

21. A spring system is set up as follows: a platform with a weight of 10 N is on top of two springs, each with spring constant 75N/m. On top of the platform is a third spring with spring constant 75N/m. If a ball with a weight of 5.0 N is then fastened to the top of the third spring and then slowly lowered, by how much does the height of the spring system change?



(A) 0.033 m
(B) 0.067 m
(C) 0.100 m
(D) 0.133 m
(E) 0.600 m

22. The softest audible sound has an intensity of $I_0 = 10^{-12}$ W/m^2. In terms of the fundamental units of kilograms, meters, and seconds, this is equivalent to

(A) $I_0 = 10^{-12}$ kg/s^3
(B) $I_0 = 10^{-12}$ kg/s
(C) $I_0 = 10^{-12}$ kg^{2}m/s
(D) $I_0 = 10^{-12}$ kg^{2}m/s^2
(E) $I_0 = 10^{-12}$ kg/m $\cdot$ s^3

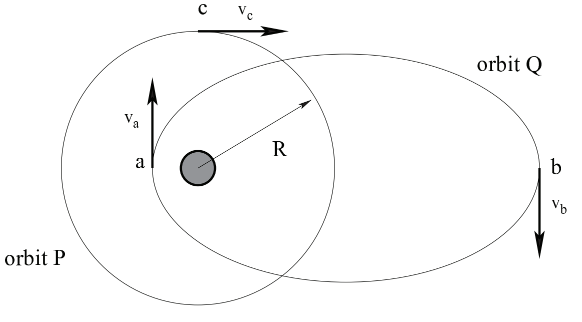
23. Which of the following sets of equipment **cannot** be used to measure the local value of the acceleration due to gravity (g)?

    (A) A spring scale (which reads in force units) and a known mass.
    (B) A rod of known length, an unknown mass, and a stopwatch.
    (C) An inclined plane of known inclination, several carts of different known masses, and a stopwatch.
    (D) A launcher which launches projectiles at a known speed, a projectile of known mass, and a meter stick.
    (E) A motor with a known output power, a known mass, a piece of string of unknown length, and a stopwatch.

24. Three point masses m are attached together by identical springs. When placed at rest on a horizontal surface the masses form a triangle with side length l. When the assembly is rotated about its center at angular velocity ω, the masses form a triangle with side length $2l$. What is the spring constant k of the springs?

    (A) $2m\omega^2$
    (B) $\frac{2}{\sqrt{3}}m\omega^2$
    (C) $\frac{2}{3}m\omega^2$
    (D) $\frac{1}{\sqrt{3}}m\omega^2$
    (E) $\frac{1}{3}m\omega^2$

25. Consider the two orbits around the sun shown below. Orbit P is circular with radius R, orbit Q is elliptical such that the farthest point b is between $2R$ and $3R$, and the nearest point a is between $R/3$ and $R/2$. Consider the magnitudes of the velocity of the circular orbit v_c, the velocity of the comet in the elliptical orbit at the farthest point v_b, and the velocity of the comet in the elliptical orbit at the nearest point v_a. Which of the following rankings is correct?



    (A) $v_b > v_c > 2v_a$
    (B) $2v_c > v_b > v_a$
    (C) $10v_b > v_a > v_c$
    (D) $v_c > v_a > 4v_b$
    (E) $2v_a > \sqrt{2}v_b > v_c$

Answers, Problem Difficulty, and Topics

ToC

2012.1	D	⋆	linear motion
2012.2	A	⋆ ⋆	projectile motion
2012.3	C	⋆	center of mass
2012.4	B	⋆	conservation of linear momentum
2012.5	D	⋆ ⋆	collisions, energy
2012.6	B	⋆	linear motion
2012.7	E	⋆	linear motion
2012.8	B	⋆ ⋆	kinetic friction, springs, work, energy
2012.9	C	⋆ ⋆	gravitation
2012.10	D	⋆ ⋆ ⋆	rolling motion, conservation of energy
2012.11	B	⋆ ⋆	kinetic friction, tension
2012.12	D	⋆ ⋆	forces, equilibrium
2012.13	E	⋆ ⋆	kinetic energy, work
2012.14	A	⋆ ⋆	torque
2012.15	A	⋆ ⋆	power
2012.16	E	⋆ ⋆	springs, oscillations
2012.17	A	⋆	data analysis
2012.18	B	⋆	springs, oscillations
2012.19	D	⋆ ⋆ ⋆	fluids, work, power
2012.20	E	⋆ ⋆	Archimedes' Principle
2012.21	C	⋆ ⋆	springs, Hooke's Law
2012.22	A	⋆	dimensional analysis
2012.23	C	⋆ ⋆	thought experiments, power
2012.24	C	⋆ ⋆ ⋆	springs, Hooke's Law
2012.25	C	⋆ ⋆ ⋆	Kepler's Laws

Solutions

ToC

2012.1. Let us denote the height of the faucet above the sink by H and the time a drop needs to fall from the faucet to the sink by T. We can write

$$H = \frac{gT^2}{2}.$$

Time T after a drop started falling, another one is already falling, and the third is about to start falling. Because drops form at regular intervals, the time needed for a new drop to form is

$$t = T/2.$$

How much did the drop that is currently in the air fall already? It started falling time t ago, so the answer is

$$y = \frac{gt^2}{2} = \frac{gT^2}{8} = \frac{H}{4}.$$

How high is it above the sink at the moment?

$$h = H - y = H - \frac{H}{4} = \frac{3H}{4} = 7.5 \text{ cm},$$

so the answer is D.

2012.2. The maximum horizontal range is given by

$$R = \frac{v_0^2 \sin 2\theta}{g},$$

while the maximum vertical range is given by

$$H = \frac{v_0^2 \sin^2 \theta}{2g}.$$

The condition $H > R$ gives us (using $\sin 2\theta = 2 \sin\theta \cos\theta$)

$$\frac{v_0^2 \sin^2 \theta}{2g} > \frac{v_0^2 \sin 2\theta}{g} \quad \Rightarrow \quad \tan\theta > 4 \quad \Rightarrow \quad \theta_{min} \approx 76^o,$$

and the answer is A.

2012.3. In general, an object will *not* topple as long as its centroid (center of mass) is above the contact area between the object and the ground. From this we find that an equilateral triangle will begin to topple when the incline is at 60^o with the horizontal, so the answer is C.

2012.4. Since there are no external forces, the linear momentum is conserved in this explosion. The linear momentum in the reference system of the particle is $p_i = 0$, and so after the explosion it has to be $p_f = p_i = 0$. If the two perpendicular velocities are in directions of north and east, their vector sum will have magnitude $v\sqrt{2}$ and will be pointed in the northeast direction. The velocity of the third particle must be exactly the same in terms of the magnitude (which answers the question) and must have the opposite direction, i.e., southwest (not part of the question). Therefore, the answer is B.

2012.5. Let us denote $m_1 = 12$ kg, $v_1 = 4$ m/s, $m_2 = 6$ kg, and $v_2 = -2$ m/s (note the two velocities have opposite signs to indicate that the two object move in opposite directions). After the collision, objects travel together (i.e., stick together), so this is a perfectly inelastic collision and only linear momentum is conserved:

$$(m_1 + m_2)w = m_1 v_1 + m_2 v_2 \quad \Rightarrow \quad w = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

The loss of kinetic energy is

$$\Delta E_k = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - \frac{(m_1 + m_2)w^2}{2}.$$

While we could do our calculations at this point, it is nice to do the algebra all the way through and get

$$\Delta E_k = \frac{m_1 m_2 (v_1 - v_2)^2}{2(m_1 + m_2)} = 72 \text{ J},$$

so the answer is D.

2012.6. With $x = 200$ m, $v_1 = 25$ m/s, $v_2 = 55$ m/s, and t denoting the collision time, we can write

$$x = v_1 t - \frac{gt^2}{2} + v_2 t + \frac{gt^2}{2} = (v_1 + v_2)t,$$

so

$$t = \frac{x}{v_1 + v_2} = 2.5 \text{ s},$$

and the answer is B.

2012.7. Using the same notation as in the previous problem, we can find the answer from

$$h = v_2 t + \frac{gt^2}{2} = 168.75 \text{ m},$$

and the best answer is E.

2012.8. The kinetic energy of the block is converted into the potential energy of the spring and heat due to the work of the force of kinetic friction, therefore

$$\frac{mv^2}{2} = \frac{kx^2}{2} + \mu_k mgx.$$

This is a quadratic equation in x, which after substitutions becomes

$$40x^2 + 15x - 6 = 0,$$

and has solutions

$$x_1 = 0.24 \text{ m} \quad \text{and} \quad x_2 = -0.62 \text{ m}.$$

We allow only positive solutions, so we reject x_2, and the answer is B.

2012.9. In order to be able to escape the planet's gravity starting from its surface, the initial kinetic energy of the object must be at least equal to the difference of gravitational potentials at infinity (0) and on the surface ($-G\frac{mM}{R}$), i.e.,

$$\frac{mv^2}{2} = G\frac{mM}{R} \quad \Rightarrow \quad v = \sqrt{2\frac{GM}{R}} = \sqrt{2gR}.$$

The last equality comes from the two expressions for the weight of the object on the surface:

$$mg = G\frac{mM}{R^2} \quad \Rightarrow \quad g = \frac{GM}{R^2}.$$

Therefore, the answer is C.

2012.10. Had these object been only sliding down the incline, their accelerations would have been $a = g\sin\theta$, where θ is the angle between the incline and the horizontal plane, and they would arrive to the finish line at the same time. However, because there is rolling without slipping, some of the potential energy difference will be going into the rotational kinetic energy, thus making the translatory motion slower than in the sliding-only case.

Approach 1: We will show that the acceleration in this experiment does not depend on the density of an object, nor on its radius or diameter, nor on its mass, but only on its shape. Let the mass of the object be m, its radius r, and its moment of inertia $I = cmr^2$, where c is the constant that depends on the object shape. In general, the value of c can be determined experimentally, while for regular geometric objects it can be determined using integration and the parallel axis theorem. For example,

- a hollow cylinder has $c = 1$
- a solid cylinder has $c = 1/2$
- a hollow sphere has $c = 2/3$
- a solid ball has $c = 2/5$
- a stick rotating around its center has $c = 1/12$
- a stick rotating around its end has $c = 1/3$

Let us determine the acceleration of an object characterized by m, r, and c. We write the equation for energies before the object starts rolling and when it traveled distance s along the slope, thus having a vertical displacement $h = s \sin\theta$:

$$E_p = E_k^{lin} + E_k^{rot} \qquad \Rightarrow \qquad mgh = \frac{mv^2}{2} + \frac{I\omega^2}{2}.$$

There is no slipping so $v = \omega r$. Also, $h = s \sin\theta$ and we know $I = cmr^2$, so

$$2gs\sin\theta = v^2 + cv^2 \qquad \Rightarrow \qquad v^2 = \frac{2gs\sin\theta}{1+c}.$$

Combining this with $v^2 = 2as$, we get

$$a = \frac{v^2}{2s} = \frac{g\sin\theta}{1+c}.$$

We see that this expression does not depend on any other properties of the body, only its shape, as characterized by c. We also see that the smaller the value of c, the greater the acceleration.

Going back to our problem, we see that the smaller the value of c, the greater will its acceleration be, therefore the shorter its time to the finish line. Since objects A, B, and D all have the same shape (solid ball, $c = 2/5$), they will all arrive at the same time, while object C (a hollow sphere, $c = 2/3$) will arrive after them. Therefore, the answer is D.

Approach 2: This problem can be solved faster if you already know that the time to the finish line will only depend on the shape of the object. If the moment of inertia of a body is $I = cmr^2$, the ratio between the translatory and rotational kinetic energy is (we use the no-slipping condition $v = \omega r$)

$$\frac{E_k^{rot}}{E_k^{transl}} = \frac{\frac{I\omega^2}{2}}{\frac{mv^2}{2}} = \frac{\frac{cmr^2v^2}{r^2}}{mv^2} = c.$$

Since the sum of E_k^{rot} and E_k^{transl} is constant (the difference between the initial and final potential energies), i.e., $E_k^{rot} + E_k^{transl} = E_p$, objects with smaller c will have less rotational kinetic energy (and thus more translatory kinetic energy) than objects with greater c. Therefore, rolling objects will arrive to the finish line in the order of increasing values of c.

In our problem, objects A, B, and D all have the same shape with $c = 2/5$, so they will all arrive at the same time, while object C has $c = 2/3$, so it will arrive after them. Therefore, the answer is D.

Note: What happens if the rolling had some slipping?

2012.11. Let θ be the angle formed by the non-horizontal section of the rope and the horizontal. Let also m be the mass of the box and T the tension in the rope. Then T is also the force applied on the rope by Lily. The vertical component of the force applied to the box is $T_y = T\sin\theta$, while its horizontal component is $T_x = T\cos\theta$. Therefore, the force of friction is

$$F_f = \mu(mg - T_y) = \mu(mg - T\sin\theta).$$

Let us now consider all available answers one by one.

(A) Let us determine Lily's force T so that its horizontal component matches the kinetic friction for any angle θ:

$$T_x = F_f \quad \Rightarrow \quad T\cos\theta = \mu(mg - T\sin\theta) \quad \Rightarrow \quad T = \frac{\mu mg}{\cos\theta + \mu\sin\theta}.$$

Because θ changes over time, force T cannot be constant, so this answer choice is out.

(B) From the expression $F_f = \mu(mg - T\sin\theta)$ we see that as θ increases over time, $\sin\theta$ does too, and so F_f decreases over time. This is the correct answer.

(C) The normal force of the ground is $N = mg - T_y = mg - T\sin\theta = mg\left(1 - \frac{\mu}{\cot\theta+\mu}\right)$. As θ increases over time, $\cot\theta$ decreases, and so N will decrease over time. This answer is out.

(D) The pressure is proportional to N, so it decreases too. This answer is out.

(E) The pressure is proportional to N, so it cannot be constant. This answer is out.

We see that the answer is B.

2012.12. When the system is in the final position, it looks as in Figure 1.

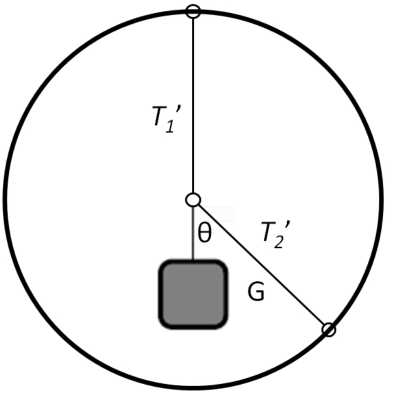


Figure 1: Illustration for Problem 2012.12.

From the equilibrium of forces, we can write

$$\text{vertically:} \qquad T_1' - mg - T_2' \cos\theta = 0$$

$$\text{horizontally:} \qquad T_2' \sin\theta = 0.$$

From this we have

$$T_2' = 0 \qquad \Rightarrow \qquad \text{so the correct answer is D.}$$

2012.13. Initially, the cart is moving to the left and is being slowed down by the force of 5 N. How far left will it go? Let us denote the cart's leftmost position by x. The cart's initial kinetic energy is $E_{k0} = \frac{mv_0^2}{2} = 18$ J and it is being reduced by the work of the opposing force. That gives us how far to the left the cart will go:

$$Fx = -E_{k0} \qquad \Rightarrow \qquad x = -3.6 \text{ m}.$$

Next, the cart moves to the right from rest at $x = -3.6$ m and will achieve v_{max} at the point where $F(x) = 0$, i.e., at $x = 6$ m. There the kinetic energy will be equal to the area under the graph of $F(x)$ between $x = -3.6$ m and $x = 6$ m:

$$E_{k1} = (1 + 3.6) \cdot 5 + \frac{1}{2} \cdot 5 \cdot 5 \text{ J} = 35.5 \text{ J}.$$

Finally,

$$E_{k1} = \frac{mv_{max}^2}{2} \qquad \Rightarrow \qquad v_{max} = 4.2 \text{ m/s} \qquad \Rightarrow \qquad \text{the answer is E.}$$

2012.14. We can choose an arbitrary center of rotation to solve this problem. In order to simplify our solution, let us choose the center of rotation so that the torque due to static friction is zero. That is accomplished by picking the point of contact with the ground as the center of rotation, because the lever arm of static friction is then 0.
The magnitude of the torque due to force T around the contact point with the ground is ($\tau_1 = T \cdot 2a$). We want it to match the torque caused by the hole. To more experienced problem solvers it will be obvious that

$$\tau_2 = W \cdot \frac{2}{5}a,$$

where W is the weight of the removed material. Let us derive this for those less experienced. Let us denote the point of contact by C, the center of the cross section of the cylinder by O, and the center of the hole by M. By definition, the magnitude of torque τ_2 is

$$\tau_2 = W \cdot [CM] \cdot \sin\theta,$$

where $\theta = \angle(CO, CM)$. Since $[CM] \cdot \sin\theta = \frac{2a}{5}$, we see that indeed

$$\tau_2 = W \cdot \frac{2}{5}a.$$

Therefore,

$$\tau_1 = \tau_2 \quad \Rightarrow \quad 2aT = \frac{2aW}{5} \quad \Rightarrow \quad T = \frac{W}{5} = 3 \text{ N},$$

so the closest answer is A.

2012.15. This problem instructs us to neglect friction. It is the friction with the air that we need to neglect, not the static friction with the ground, otherwise there would be no motion up the hill.

With that ambiguity out of the way, let us look at the problem. If the car is to move up at a constant speed, the net force on it must be 0, so the force that propels the car, F, must be equal to the component of gravity parallel to the slope, $mg\sin\theta$. Therefore,

$$P = Fv_{max} = mgv_{max}\sin\theta \quad \Rightarrow \quad v_{max} = \frac{P}{mg\sin\theta},$$

so the answer is A.

2012.16. Even though this spring-mass configuration may look like a serial connection of springs, it is in fact a parallel configuration, because the lengths of both springs change the full amount of motion of the mass[1]. Therefore, the equivalent spring constant is $k = k_1 + k_2$ and the oscillation frequency is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{M}}.$$

The acceleration of the system does not affect the period (or the frequency for that matter) of a spring-mass system, only its equilibrium point, so the answer is E.

2012.17. The graph tells us that $\log T = 2\log A + 3$, therefore

$$\log T = \log A^2 + \log 10^3 = \log(1000A^2) \quad \Rightarrow \quad T = 1000A^2,$$

and the answer is A.

2012.18. As mentioned in Problem 2012.16, the period of oscillations of a spring-mass system is not affected by the acceleration of the system; therefore, the right answer is B.

2012.19. Let us consider the volume V of water being pumped during time Δt. Let us denote the mass of that volume of water by m and by S the cross-sectional area of the pipe. Then

$$m = \rho V = \rho S v \Delta t = \rho r^2 \pi v \Delta t.$$

The work done by the pump is used to accelerate the water to speed v and to lift it up by height h:

$$W = P\Delta t = mgh + \frac{mv^2}{2}.$$

Therefore,

$$P\Delta t = mgh + \frac{mv^2}{2} = \rho r^2 \pi v \Delta t \left(gh + \frac{v^2}{2}\right),$$

so

$$r = \sqrt{\frac{P}{\rho \pi v \left(gh + \frac{v^2}{2}\right)}} = 9.1 \text{ cm} \quad \Rightarrow \quad \text{the closest answer is D.}$$

[1] Another way to see this is to note that the force on the mass is the sum of the two spring forces. For springs in a series configuration, the force on the object is equal to the forces of individual springs, while the length changes of the springs add to the mass displacement.

Note 1: An alternative way to get this result is to use Bernoulli's Principle to express the gauge pressure of water due to the engine: $p = \rho g h + \rho v^2/2$. Using this in $P = Fv = pSv$, we get

$$P = \rho \left(gh + \frac{v^2}{2} \right) r^2 \pi v.$$

Note 2: A common mistake made in similar problems is to write $P = Fv$ and to say that (here is the mistake) $F = mg$. In reality, the force exerted by the pump is greater than mg because the water is not only lifted up but also accelerated from 0 to v. Because of the answer choices in this particular problem, even this wrong approach leads to the right answer. In general, we have to be careful about this.

2012.20. Due to the force of buoyancy acting on the half-submerged block of wood, the scale on the bottom will show a higher value while the scale on the top will show a lower value. Both will show the mass difference corresponding to the force of buoyancy, $\Delta M = B/g$. If ρ_0 is the density of water, ρ the density of wood, and V the total volume of the block of wood, the buoyancy is

$$B = \rho_0 \frac{V}{2} g = \frac{\rho_0 M_2 g}{2\rho}.$$

The mass corresponding to buoyancy is the mass of the displaced water:

$$\Delta M = B/g = \frac{\rho_0 M_2}{2\rho} = 10 \text{ kg},$$

so the new readings will be 55 kg and 2 kg, and the answer is E.

2012.21. First note that the weight of the platform is not relevant for this problem. We can replace the three springs in this problem with an equivalent spring obtained as a serial connection of: (a) two bottom springs in parallel and (b) the top spring. The equivalent spring constant k' can be determined from

$$\frac{1}{k'} = \frac{1}{k} + \frac{1}{2k} = \frac{3}{2k} \quad \Rightarrow \quad k' = \frac{2k}{3}.$$

The weight W of the ball will compress the equivalent spring by

$$\Delta x = \frac{W}{k'} = \frac{3W}{2k} = 0.1 \text{ m},$$

so the answer is C.

2012.22. Knowing that $P = \frac{E}{t} = \frac{Fs}{t} = \frac{mas}{t}$, for the unit of power, the watt, we can write

$$[P] = \mathrm{W} = \frac{\mathrm{kg \cdot m/s^2 \cdot m}}{\mathrm{s}} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^3}}.$$

Therefore $\mathrm{W/m^2} = \mathrm{kg/s^3}$, and the answer is A.

2012.23. Let us explain how answers A, B, D, and E can be used to measure g locally:

(A) With the scale reading of the weight W and with a known mass m, we can use the formula $W = mg$ to calculate $g = W/m$.

(B) Depending on whether the rod has mass or not, we can use the formula for physical or simple pendulum, respectively, to determine g.

(C) Not possible because in order to get g from the time measurement, we also need to know the distance the carts traveled in that time, which we do not have a way of getting.

(D) With this setup we can launch the projectile vertically up and measure the maximum height it reaches (the projectile mass is not needed here). Then $v^2 = 2gh \;\Rightarrow\; g = \frac{v^2}{2h}$.

(E) We are not sure if the following is what the author of the problem had in mind in this part, but this is what can be used to reason that the experiment is possible:

1. First, we use the stopwatch to measure time T_1 needed for the motor to lift the mass by the (unknown) length of string L. For this we can write
$$PT_1 = mgL.$$
2. Next, we can drop the mass (or the motor!)[2] from height L and measure the fall time T_2 with the stopwatch. For this we can write
$$L = \frac{gT_2^2}{2}.$$

Combining the two equations, we get $g = \frac{\sqrt{2PT_1/m}}{T_2}$.

We see that the only impossible setup is C.

[2]Our inspiration for this experiment came from an old electrical engineering joke asking how to measure the height of a building if we are given a stopwatch, the value of g, and a really expensive oscilloscope. The answer (of course) is to throw the oscilloscope from the top of the building, measure its fall time, and calculate the height as $h = gt^2/2$.

2012.24. We will solve this problem using two different approaches. The problem setup is illustrated in Figure 2:

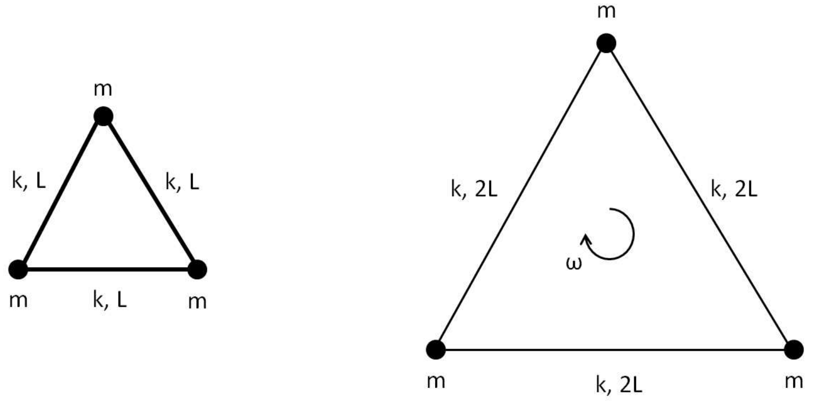


Figure 2: Illustration for Problem 2012.24.

Approach 1: Our three springs are getting extended because of the centrifugal force $F_c = m\omega^2 R$, where R is the radius of rotation:

$$R = \frac{2}{3} \cdot \frac{\sqrt{3}}{2}(2L) = \frac{2L\sqrt{3}}{3}.$$

What is the magnitude of the centrifugal force's component that is acting in the direction of a spring at a vertex? To be specific, let us consider the spring between the mass at the top of Figure 2 and the mass on the bottom left. Let us focus on what happens at the top mass.

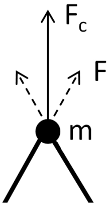


Figure 3: Illustration for Problem 2012.24.

From Figure 3 we see[3] that $2F\cos 30^o = F_c \;\Rightarrow\; F = F_c/\sqrt{3}$. The same analysis holds for the other end of our spring, where another force with magnitude $F = F_c/\sqrt{3}$ acts to stretch our spring. Recall that in Problem 2011.9 we discussed a similar situation, with two equal magnitude forces stretching a

[3]Note that this decomposition is *not* a decomposition into two orthogonal vectors, therefore $F \neq F_c \cos 30^o$.

spring. We concluded that this is equivalent to one of the ends of the spring being attached to a fixed wall. Therefore, we can write

$$k(2L - L) = F = F_c/\sqrt{3} = \frac{m\omega^2 R}{\sqrt{3}} = \frac{2m\omega^2 L}{3}.$$

Cancelling L on both sides, we get

$$k = \frac{2m\omega^2}{3} \quad \Rightarrow \quad \text{the answer is C.}$$

Approach 2: In this approach we transform the triangle (or delta, or Δ) configuration of springs into a star (or Y) configuration. The spring constant of each spring in the equivalent star configuration is $k' = 3k$ (you may want to remember this as you may need it in a future competition; the more general transformation formulas for arbitrary spring constants in the triangle are analogous to the "star-delta transformation" for capacitors in electric circuits).

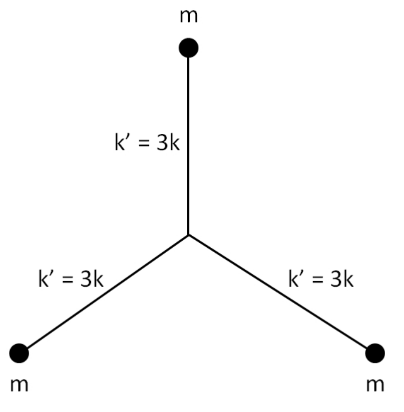


Figure 4: Illustration for Problem 2012.24.

The original, unextended length of the bands in star configuration is

$$D = \frac{2}{3}L\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}L$$

and it doubles when the system rotates, so $D' = 2D = \frac{2\sqrt{3}}{3}L$. The centripetal force on each mass is the spring restoration force of elastic bands, so

$$m\omega^2 D' = k'(D' - D) \quad \Rightarrow \quad 2m\omega^2 D = 3kD \quad \Rightarrow \quad k = \frac{2}{3}m\omega^2,$$

and the answer is C.

Note: We can derive the transformation $k' = 3k$ by comparing the results of the two approaches for this problem. In Approach 1 we found that

$$k = \frac{2m\omega^2}{3}.$$

If in Approach 2 we assume that $k' = \alpha k$ for some real constant α, we find that

$$k = \frac{2m\omega^2}{\alpha}.$$

The two approaches must give the same result, therefore $\alpha = 3$ and so $k' = 3k$.

2012.25. Answers A, B, D, E can be eliminated based on the conservation of angular momentum (Kepler's Second Law) and the conservation of energy. In particular,

(A) says $v_b > 2v_a$, but that is obviously wrong, because from conservation of angular momentum $v_b < v_a$.

(B) says $v_b > v_a$, again wrong.

(C) will be the correct one.

(D) says $v_c > v_a$, but that is wrong because A is the perihelion of an ellipse, so v_a is greater than it would be if A was on a circular orbit[4] and the speed on that circular orbit is greater than v_c because (from equality of centripetal and gravitational force)

$$v = \sqrt{GM/r} \quad \text{and} \quad r_c > r_a.$$

(E) says that $\sqrt{2}v_b > v_c$, which is wrong, and here is why. For a circular orbit,

$$\frac{mv_c^2}{R} = G\frac{mM}{R^2} \quad \Rightarrow \quad v_c^2 = \frac{GM}{R}.$$

For an elliptical orbit, we can use conservation of angular momentum with $r_a = kR$ (where $k \in (\frac{1}{3}, \frac{1}{2})$) and $r_b = KR$ (where $K \in (2, 3)$) to get

$$mv_a kR = mv_b KR \quad \Rightarrow \quad v_a = \frac{K}{k}v_b.$$

[4] Why is this true? Because the speed on a circular orbit is the *first cosmic speed* for the Sun and that orbit radius, and as you increase the speed in that point, that point becomes a perihelion of an elliptical orbit, with bigger and bigger aphelion, until you reach the *second cosmic speed* for the Sun and the orbit becomes parabolic, then hyperbolic, and the object leaves the Solar system.

Note that the factor $\frac{K}{k}$ can be between 4 and 9, so

$$4v_b < v_a < 9v_b.$$

The conservation of energy on the elliptical orbit becomes

$$-G\frac{mM}{kR} + \frac{mv_a^2}{2} = -G\frac{mM}{KR} + \frac{mv_b^2}{2}.$$

With $v_c^2 = \frac{GM}{R}$, this yields

$$2v_c^2 = \frac{K}{k}(K+k)v_b^2.$$

The factor $\frac{K}{k}(K+k)$ can be between 10 and 30, so

$$\sqrt{5}v_b < v_c < \sqrt{15}v_b.$$

Therefore, v_c cannot be less than $\sqrt{2}v_b$, so E must be wrong too.

The only answer that is consistent with everything we figured out about the relationships between v_a, v_b, and v_c is C.

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Year 2013

$\boldsymbol{F = ma}$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP CEE 2013

2013 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers will result in a deduction of $\frac{1}{4}$ point. There is no penalty for leaving an answer blank.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2013.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Contributors to this year's exam include David Fallest, Jiajia Dong, Paul Stanley, Warren Turner, Qiuzi Li, and former US Team members Marianna Mao, Andrew Lin, Steve Byrnes.

1. An observer stands on the side of the front of a stationary train. When the train starts moving with constant acceleration, it takes 5 seconds for the first car to pass the observer. How long will it take for the 10th car to pass?

   (A) 1.07s
   (B) 0.98s
   (C) 0.91s
   (D) 0.86s
   (E) 0.81s

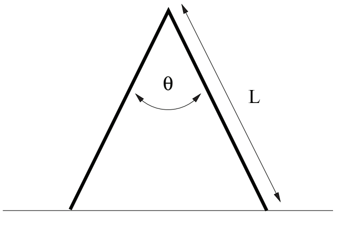
2. Jordi stands 20 m from a wall and Diego stands 10 m from the same wall. Jordi throws a ball at an angle of 30° above the horizontal, and it collides elastically with the wall. How fast does Jordi need to throw the ball so that Diego will catch it? Consider Jordi and Diego to be the same height, and both are on the same perpendicular line from the wall.

   (A) 11 m/s
   (B) 15 m/s
   (C) 19 m/s
   (D) 30 m/s
   (E) 35 m/s

3. Tom throws a football to Wes, who is a distance l away. Tom can control the time of flight t of the ball by choosing any speed up to v_{max} and any launch angle between 0° and 90°. Ignore air resistance and assume Tom and Wes are at the same height. Which of the following statements is **incorrect**?

   (A) If $v_{max} < \sqrt{gl}$, the ball cannot reach Wes at all.
   (B) Assuming the ball can reach Wes, as v_{max} increases with l held fixed, the minimum value of t decreases.
   (C) Assuming the ball can reach Wes, as v_{max} increases with l held fixed, the maximum value of t increases.
   (D) Assuming the ball can reach Wes, as l increases with v_{max} held fixed, the minimum value of t increases.
   (E) Assuming the ball can reach Wes, as l increases with v_{max} held fixed, the maximum value of t increases.

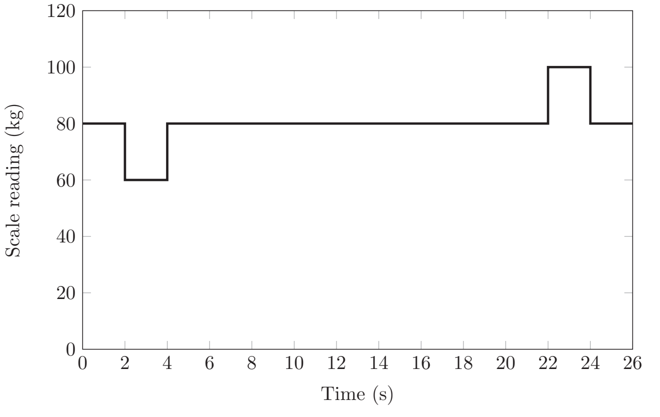
4. The sign shown below consists of two uniform legs attached by a frictionless hinge. The coefficient of friction between the ground and the legs is μ. Which of the following gives the maximum value of θ such that the sign will not collapse?



   (A) $\sin\theta = 2\mu$
   (B) $\sin\theta/2 = \mu/2$
   (C) $\tan\theta/2 = \mu$
   (D) $\tan\theta = 2\mu$
   (E) $\tan\theta/2 = 2\mu$

The following information applies to questions 5 and 6

A student steps onto a stationary elevator and stands on a bathroom scale. The elevator then travels from the top of the building to the bottom. The student records the reading on the scale as a function of time.



5. At what time(s) does the student have maximum downward velocity?

   (A) At all times between 2 s and 4 s
   (B) At 4 s only
   (C) At all times between 4 s and 22 s
   (D) At 22 s only
   (E) At all times between 22 s and 24 s

6. How tall is the building?

   (A) 50 m
   (B) 80 m
   (C) 100 m
   (D) 150 m
   (E) 400 m

7. A light car and a heavy truck have the same momentum. The truck weighs ten times as much as the car. How do their kinetic energies compare?

   (A) The truck's kinetic energy is larger by a factor of 100
   (B) They truck's kinetic energy is larger by a factor of 10
   (C) They have the same kinetic energy
   (D) The car's kinetic energy is larger by a factor of 10
   (E) The car's kinetic energy is larger by a factor of 100

The following information applies to questions 8 and 9

A truck is initially moving at velocity v. The driver presses the brake in order to slow the truck to a stop. The brake applies a constant force F to the truck. The truck rolls a distance x before coming to a stop, and the time it takes to stop is t.

8. Which of the following expressions is equal the initial kinetic energy of the truck (i.e. the kinetic energy before the driver starts braking)?

   (A) Fx
   (B) Fvt
   (C) Fxt
   (D) Ft
   (E) Both (a) and (b) are correct

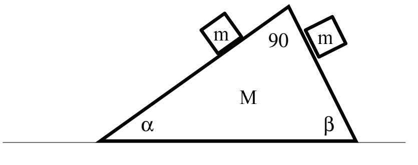
9. Which of the following expressions is equal the initial momentum of the truck (i.e. the momentum before the driver starts braking)?

   (A) Fx
   (B) $Ft/2$
   (C) Fxt
   (D) $2Ft$
   (E) $2Fx/v$

10. Which of the following can be used to distinguish a solid ball from a hollow sphere of the same radius and mass?

   (A) Measurements of the orbit of a test mass around the object.
   (B) Measurements of the time it takes the object to roll down an inclined plane.
   (C) Measurements of the tidal forces applied by the object to a liquid body.
   (D) Measurements of the behavior of the object as it floats in water.
   (E) Measurements of the force applied to the object by a uniform gravitational field.

11. A right-triangular wooden block of mass M is at rest on a table, as shown in figure. Two smaller wooden cubes, both with mass m, initially rest on the two sides of the larger block. As all contact surfaces are frictionless, the smaller cubes start sliding down the larger block while the block remains at rest. What is the normal force from the system to the table?



   (A) $2mg$
   (B) $2mg + Mg$
   (C) $mg + Mg$
   (D) $Mg + mg(\sin\alpha + \sin\beta)$
   (E) $Mg + mg(\cos\alpha + \cos\beta)$

12. A spherical shell of mass M and radius R is completely filled with a frictionless fluid, also of mass M. It is released from rest, and then it rolls without slipping down an incline that makes an angle θ with the horizontal. What will be the acceleration of the shell down the incline just after it is released? Assume the acceleration of free fall is g.

    The moment of inertia of a thin shell of radius r and mass m about the center of mass is $I = \frac{2}{3}mr^2$; the moment of inertia of a solid sphere of radius r and mass m about the center of mass is $I = \frac{2}{5}mr^2$.

    (A) $a = g \sin\theta$
    (B) $a = \frac{3}{4}g \sin\theta$
    (C) $a = \frac{1}{2}g \sin\theta$
    (D) $a = \frac{3}{8}g \sin\theta$
    (E) $a = \frac{3}{5}g \sin\theta$

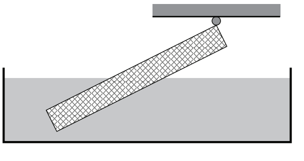
13. There is a ring outside of Saturn. In order to distinguish if the ring is actually a part of Saturn or is instead part of the satellites of Saturn, we need to know the relation between the velocity v of each layer in the ring and the distance R of the layer to the center of Saturn. Which of the following statements is correct?

    (A) If $v \propto R$, then the layer is part of Saturn.
    (B) If $v^2 \propto R$, then the layer is part of the satellites of Saturn.
    (C) If $v \propto 1/R$, then the layer is part of Saturn.
    (D) If $v^2 \propto 1/R$, then the layer is part of Saturn.
    (E) If $v \propto R^2$, then the layer is part of the satellites of Saturn.

14. A cart of mass m moving at 12 m/s to the right collides elastically with a cart of mass 4.0 kg that is originally at rest. After the collision, the cart of mass m moves to the left with a velocity of 6.0 m/s. Assuming an elastic collision in one dimension only, what is the velocity of the center of mass (v_{cm}) of the two carts *before* the collision?

    (A) $v_{\text{cm}} = 2.0$ m/s
    (B) $v_{\text{cm}} = 3.0$ m/s
    (C) $v_{\text{cm}} = 6.0$ m/s
    (D) $v_{\text{cm}} = 9.0$ m/s
    (E) $v_{\text{cm}} = 18$ m/s

15. A uniform rod is partially in water with one end suspended, as shown in figure. The density of the rod is 5/9 that of water. At equilibrium, what portion of the rod is above water?



    (A) 0.25
    (B) 0.33
    (C) 0.5
    (D) 0.67
    (E) 0.75

16. *Inspired by a problem from the 2012 International Physics Olympiad, Estonia.*

A very large number of small particles forms a spherical cloud. Initially they are at rest, have uniform mass density per unit volume ρ_0, and occupy a region of radius r_0. The cloud collapses due to gravitation; the particles do not interact with each other in any other way.

How much time passes until the cloud collapses fully? (The constant 0.5427 is actually $\sqrt{\frac{3\pi}{32}}$.)

(A) $\frac{0.5427}{{r_0}^2\sqrt{G\rho_0}}$

(B) $\frac{0.5427}{r_0\sqrt{G\rho_0}}$

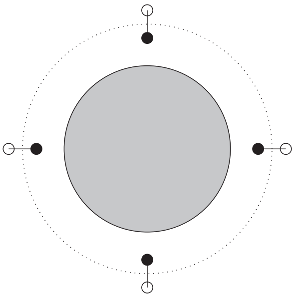
(C) $\frac{0.5427}{\sqrt{r_0}\sqrt{G\rho_0}}$

(D) $\frac{0.5427}{\sqrt{G\rho_0}}$

(E) $\frac{0.5427}{\sqrt{G\rho_0}}r_0$

17. Two small, equal masses are attached by a lightweight rod. This object orbits a planet; the length of the rod is smaller than the radius of the orbit, but not negligible. The rod rotates about its axis in such a way that it remains vertical with respect to the planet.

- Is there a force in the rod? If so, is it tension or compression?
- Is the equilibrium stable, unstable, or neutral with respect to a small perturbation in the angle of the rod? (Assume this perturbation maintains the rate of rotation, so that in the co-rotating frame the rod is still stationary but at an angle to the vertical.)



(A) There is no force in the rod; the equilibrium is neutral.

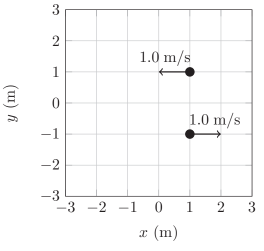
(B) The rod is in tension; the equilibrium is stable.

(C) The rod is in compression; the equilibrium is stable.

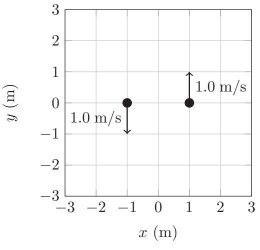
(D) The rod is in tension; the equilibrium is unstable.

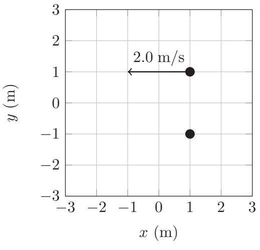
(E) The rod is in compression; the equilibrium is unstable.

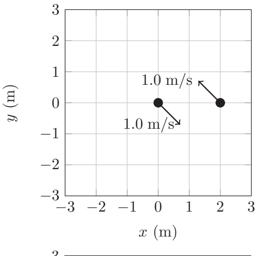
18. Two point particles, each of mass 1 kg, begin in the state shown below.

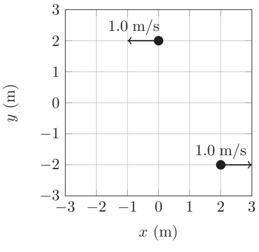


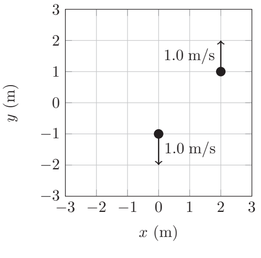
The system evolves through internal forces only. Which of the following could be the state after some time has passed?

(A)


(B)


(C)


(D)


(E)


The following information applies to questions 19, 20, and 21.

A simple pendulum experiment is constructed from a point mass m attached to a pivot by a massless rod of length L in a constant gravitational field. The rod is released from an angle $\theta_0 < \pi/2$ at rest and the period of motion is found to be T_0. Ignore air resistance and friction.

19. At what angle θ_g during the swing is the tension in the rod the greatest?

    (A) The tension is the greatest at the point $\theta_g = \theta_0$.
    (B) The tension is the greatest at the point $\theta_g = 0$.
    (C) The tension is the greatest at an angle θ_g with $0 < \theta_g < \theta_0$.
    (D) The tension is constant.
    (E) None of the above is true for all values of θ_0 with $0 < \theta_0 < \pi/2$.

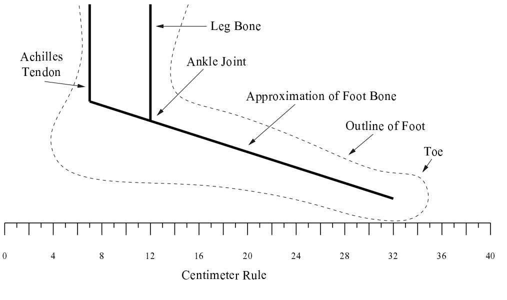
20. What is the maximum value of the tension in the rod?

    (A) mg
    (B) $2mg$
    (C) $mL\theta_0/{T_0}^2$
    (D) $mg \sin \theta_0$
    (E) $mg(3 - 2\cos\theta_0)$

21. The experiment is repeated with a new pendulum with a rod of length $4L$, using the same angle θ_0, and the period of motion is found to be T. Which of the following statements is correct?

    (A) $T = 2T_0$ regardless of the value of θ_0.
    (B) $T > 2T_0$ with $T \approx 2T_0$ if $\theta_0 \ll 1$.
    (C) $T < 2T_0$ with $T \approx 2T_0$ if $\theta_0 \ll 1$.
    (D) $T > 2T_0$ for some values of θ_0 and $T < 2T_0$ for other values of θ_0.
    (E) T_0 and T are undefined because the motion is not periodic unless $\theta_0 \ll 1$.

22. A simplified model on the foot is shown. When a student of mass $m = 60$ kg stands on a single toe, the tension T in the Achilles Tendon is closest to



    (A) $T = 600$ N
    (B) $T = 1200$ N
    (C) $T = 1800$ N
    (D) $T = 2400$ N
    (E) $T = 3000$ N

The following information applies to questions 23 and 24

A man with mass m jumps off of a high bridge with a bungee cord attached to his ankles. The man falls through a maximum distance H at which point the bungee cord brings him to a momentary rest before he bounces back up. The bungee cord is perfectly elastic, obeying Hooke's force law with a spring constant k, and stretches from an original length of L_0 to a final length $L = L_0 + h$. The maximum tension in the Bungee cord is four times the weight of the man.

23. Determine the spring constant k.

    (A) $k = \frac{mg}{h}$

    (B) $k = \frac{2mg}{h}$

    (C) $k = \frac{mg}{H}$

    (D) $k = \frac{4mg}{H}$

    (E) $k = \frac{8mg}{H}$

24. Find the maximum extension of the bungee cord h.

    (A) $h = \frac{1}{2}H$

    (B) $h = \frac{1}{4}H$

    (C) $h = \frac{1}{5}H$

    (D) $h = \frac{2}{5}H$

    (E) $h = \frac{1}{8}H$

25. A box with weight W will slide down a 30° incline at constant speed under the influence of gravity and friction alone. If instead a horizontal force P is applied to the box, the box can be made to move *up* the ramp at constant speed. What is the magnitude of P?

    (A) $P = W/2$

    (B) $P = 2W/\sqrt{3}$

    (C) $P = W$

    (D) $P = \sqrt{3}W$

    (E) $P = 2W$

Answers, Problem Difficulty, and Topics

ToC

2013.1	E	$\star$	linear motion
2013.2	C	$\star$	projectile motion
2013.3	E	$\star$	projectile motion
2013.4	E	$\star\star$	torque, equilibrium, static friction
2013.5	C	$\star$	linear motion, forces
2013.6	C	$\star$	linear motion
2013.7	D	$\star$	linear momentum, kinetic energy
2013.8	A	$\star$	work, kinetic energy
2013.9	E	$\star$	linear momentum, kinetic energy
2013.10	B	$\star\star$	moment of inertia, rolling motion
2013.11	C	$\star\star$	forces
2013.12	B	$\star\star$	rolling motion, torque, conservation of energy
2013.13	A	$\star$	rotational motion
2013.14	B	$\star\star$	collisions
2013.15	D	$\star\star\star$	Archimedes' Principle, torque
2013.16	D	$\star\star$	dimensional analysis
2013.17	B	$\star\star\star$	rotating reference frame, equilibrium
2013.18	E	$\star$	conservation of linear momentum
2013.19	B	$\star$	simple pendulum, tension
2013.20	E	$\star\star$	simple pendulum, tension
2013.21	A	$\star\star\star$	simple pendulum, oscillations
2013.22	D	$\star$	torque, tension, equilibrium
2013.23	E	$\star$	springs, energy, Hooke's Law
2013.24	A	$\star$	springs, energy, Hooke's Law
2013.25	D	$\star\star$	kinetic friction

Solutions

ToC

2013.1. Let the acceleration of the train be a and the length of one train car be ℓ. Also, consider the train to begin moving at time $t = 0$, and let t_k be the time at which the k-th car passes the observer. In particular, from the problem statement, we are given that $t_1 = 5$ s, and we wish to compute $t_{10} - t_9$. Recall that since the train starts from rests and accelerates at constant velocity, the displacement is given by

$$x(t) = \frac{1}{2}at^2.$$

When the k-th train car passes the observer, the train will have moved a distance $k\ell$. In other words,

$$x(t_k) = \frac{1}{2}at_k^2 = k\ell \quad \Rightarrow \quad t_k = \sqrt{2k\ell/a} = t_1\sqrt{k}.$$

We can then compute

$$t_{10} - t_9 = t_1(\sqrt{10} - \sqrt{9}) = (5\,\mathrm{s})(0.162) = 0.81\,\mathrm{s} \quad \Rightarrow \quad \text{the answer is E.}$$

2013.2. Let v be the initial speed of the ball and $\theta = 30^\circ$ be its initial angle. Since the wall is stationary and vertical, the ball bounces back at the same angle after it hits the wall. Effectively, it is reflected about the wall. With this interpretation, the situation is equivalent to Jordi throwing a ball to Diego's reflection on the other side of the wall, which is $d = 10\,\mathrm{m} + 20\,\mathrm{m} = 30\,\mathrm{m}$ away. Using the range equation, we see that

$$d = \frac{v^2 \sin 2\theta}{g},$$

$$v = \sqrt{\frac{gd}{\sin 2\theta}} = \sqrt{\frac{(10\,\mathrm{m/s^2})(30\,\mathrm{m})}{\sin 60^\circ}} \approx 19 \text{ m/s}.$$

Therefore, the answer is C.

2013.3. First, note that answer choice A is correct because of the *range equation*, which lets us determine the total distance traveled as

$$\ell = \frac{v^2 \sin 2\theta}{g} \leq \frac{v^2}{g} \quad \Rightarrow \quad v \geq \sqrt{g\ell}.$$

Next, let us consider the extreme cases for the time of flight. If Tom wants to minimize the time of flight, notice that the horizontal component of the ball's velocity remains constant, so Tom wants to maximize this velocity. Then, the time of flight is minimized at the maximum velocity and lowest possible angle θ such that the ball can reach Wes. Let us then examine the effects of changing the two parameters:

- As v_{max} increases, this allows for a greater velocity and lower angle, so the minimum value of t decreases.
- Conversely, as ℓ increases, a greater minimum angle is required (and thus a lower horizontal velocity), so the minimum value of t increases.

For the maximum value of t, notice that t is proportional to the vertical component of the ball's initial velocity (in particular $t = 2v \sin\theta/g$). Therefore, we should try to maximize the initial velocity of the ball, as well as its angle. If v_{max} increases, this increases the maximum value of t, but as ℓ increases, the maximum value of t decreases due to the throw requiring a less steep angle.

Therefore, increasing v_{max} expands both ends of the range for ℓ, while increasing ℓ narrows both ends of the range. Thus, the answer is E.

2013.4. Since the system is in static equilibrium, we know that the net external torque on either leg must be zero. Looking at the leg on the right, the contributing forces are

- static friction F_s, acting horizontally to the left at the contact with the floor,
- gravity, acting vertically down on all points of a leg, but equivalent to mg acting at the leg's centroid, and
- the normal force from the ground, acting vertically up at the point of contact with the floor, with magnitude mg.

If we take the torque about the hinge point, we can compute that

$$\begin{aligned}\tau &= LF_s \cos\frac{\theta}{2} + \frac{L}{2}mg\sin\frac{\theta}{2} - Lmg\sin\frac{\theta}{2} \\ &= L\cos\frac{\theta}{2}\left(F_s - \frac{1}{2}mg\tan\frac{\theta}{2}\right),\end{aligned}$$

where F_s is the force of static friction. Setting this equal to zero to account for equilibrium, we get that

$$F_s = \frac{1}{2}mg\tan\frac{\theta}{2}.$$

At the threshold angle θ, this value should equal $\mu N = \mu mg$, where N is the normal force on the leg. This gives the equation

$$\mu mg = \frac{1}{2}mg\tan\frac{\theta}{2} \quad \Rightarrow \quad 2\mu = \tan\frac{\theta}{2}.$$

Thus, the answer is E.

Note: This problem is equivalent to the problem found in many textbooks, about a stick leaning on a wall and having friction with the floor. In that problem, the angle with the vertical is $\alpha = \theta/2$, and the solution is $\mu = \frac{1}{2}\tan\alpha$. Had you done that problem before, you could solve this problem in the competition without much effort, just substituting $\theta/2$ for α. There is a version of this problem with added friction between the stick and the wall.

2013.5. The graph in this problem measures the normal force N of the scale on the student. However, it may be slightly confusing because the y-axis is in units of mass, rather than force. For example, a reading of 20 kg on the scale would actually mean a measurement of $N = (20\,\text{kg})g = 200\,\text{N}$.

From the initial reading, we can see that the student has a mass of $m = 80$ kg. In the interval from 2 s to 4 s, the normal force on the student changed by $(-20\,\text{kg})g = -200\,\text{N}$ from equilibrium, so the student accelerates downward at $a = F/m = (200\,\text{N})/(80\,\text{kg}) = -2.5\,\text{m/s}^2$. Similarly, in the interval from 22 s to 24 s, the student accelerates upward at $2.5\,\text{m/s}^2$ until once again at rest.

Then, the graph of the student's velocity looks like a trapezoid, with five piecewise linear sections:

1. Before 2 s: Constant at 0,
2. Between 2 s and 4 s: Linear with slope $-2.5\,\text{m/s}^2$,
3. Between 4 s and 22 s: Constant at $(-2.5\,\text{m/s}^2)(2\,\text{s}) = -5.0\,\text{m/s}$,
4. Between 22 s and 24 s: Linear with slope $2.5\,\text{m/s}^2$,
5. After 24 s: Constant at 0.

Therefore, maximum downward velocity is -5.0 m/s and occurs in the third section, so the answer is C.

2013.6. The height of the building is equal to the total distance x traveled by the student while in the elevator. In other words, this should be equal to the total area under the velocity-time graph described above. We can compute this with the trapezoid area formula, which gives

$$x = \frac{(b_1 + b_2)h}{2} = \frac{(18\,\text{s} + 22\,\text{s})(5.0\,\text{m/s})}{2} = 100 \text{ m}.$$

Thus, the answer is C.

2013.7. We can express kinetic energy K in terms of linear momentum p and mass m by writing

$$K = \frac{mv^2}{2} = \frac{m(p/m)^2}{2} = \frac{p^2}{2m}.$$

Then, for objects of fixed momentum, kinetic energy is inversely proportional to mass[1], i.e., $K \propto m^{-1}$. Since the truck weighs ten times more, it has one tenth the kinetic energy, so the answer is D.

2013.8. The braking does work $W = \vec{F} \cdot \vec{x} = -Fx$ on the truck, after which it ends up at rest, in a zero-kinetic energy state. Then, the initial kinetic energy must have been $K = -W = Fx$, which is answer choice A. Considering the other answers individually:

- A is correct,
- B is incorrect because the velocity of the truck decreases linearly from v to 0, so $x = vt/2$ and $Fvt = 2Fx = 2K$,
- C is incorrect because it has dimensions of $\mathrm{J \cdot s}$ (instead of J),
- D is incorrect because it has dimensions of $\mathrm{N \cdot s}$ (instead of $\mathrm{J = N \cdot m}$),
- E is incorrect because B is incorrect.

Thus, the answer is A.

2013.9. From the previous problem we know that the initial kinetic energy was $K = Fx$. We can relate this to the initial momentum p by noticing that

$$K = \frac{1}{2}mv^2 = \frac{1}{2}pv \quad \Rightarrow \quad p = \frac{2K}{v} = \frac{2Fx}{v}.$$

Thus, the answer is E.

Note: If you try the natural first step of equating impulse and momentum, you get $p = J = Ft$, which is not an answer choice. It turns out that this is equivalent to the answer given above since $x = vt/2$.

2013.10. The key differentiating principle between the two objects is that their moments of inertia I differ. If we let M be the mass and R be the radius, the hollow sphere has $I = \frac{2}{3}MR^2$, while the solid ball has $I = \frac{2}{5}MR^2$. Knowing this, we can consider each answer choice individually:

[1] If this seems counterintuitive, consider the example of a gunshot. By conservation of momentum, the momentum of the bullet is equal and opposite to the impulse on the shooter from recoil. The gun is able to put much more energy into the bullet than the recoil, as the bullet has a lower mass than the shooter.

- A is incorrect, as by Newton's shell theorem, the gravitational pull of any spherically-symmetric object on external masses is equivalent to that of a point mass.
- B seems plausible, as the rolling motion of the objects would be affected by their differing moments of inertia.
- C is incorrect, as tidal forces are due to gravity, and there should not be a difference again by the shell theorem.
- D is incorrect, as buoyancy is governed by Archimedes' Principle, which only depends on volume of fluid displaced (and thus external shape).
- E is incorrect, as both objects have the same mass.

Now we can go back to B in more detail. It turns out that if the moment of inertia of an object can be written as $I = cMR^2$, then the acceleration when rolling without slipping down an inclined plane is[2]

$$a = \frac{g \sin\theta}{1+c},$$

where θ is the angle of the incline. Since the values of c are different for the two objects, they have different accelerations and times elapsed on the incline, so the answer is B.

2013.11. Consider the system of the block and the two cubes together. The only external forces on this system are gravity, $F_g = Mg + 2mg$, and the normal force N. The vertical acceleration a of this system's center of mass can be written as

$$(M+2m)a = F_g - N.$$

We will try to find a, as this will then allow us to use the above equation to determine N.

Let a_1 be the acceleration of the cube on the angle-α ramp, and let a_2 be the acceleration of the cube on the angle-β ramp. Considering these cubes individually, the problem becomes a classic frictionless inclined plane scenario. Decomposing mg into perpendicular and parallel components yields $a_1 = g\sin\alpha$ and $a_2 = g\sin\beta$.

To find a, however, we need the vertical components of $\vec{a}_1$ and $\vec{a}_2$. Using trigonometry, these are given by $a_1 \sin\alpha$ and $a_2 \sin\beta$ respectively, so the acceleration of the center of mass is

$$a = \frac{ma_1 \sin\alpha + ma_2 \sin\beta}{M+2m} = \frac{mg(\sin^2\alpha + \sin^2\beta)}{M+2m}.$$

[2]See Problem 2012.10 for a derivation of this formula.

However, since the block is a right triangle, we have that $\alpha + \beta = 90°$, so

$$\sin^2\alpha + \sin^2\beta = \sin^2\alpha + \sin^2(90° - \alpha) = \sin^2\alpha + \cos^2\alpha = 1.$$

Then, our expression for a simplifies to $a = mg/(M + 2m)$, and combining this with the equation for N yields

$$mg = F_g - N \quad \Rightarrow \quad N = F_g - mg = mg + Mg.$$

Thus, the answer is C.

2013.12. Since the fluid is frictionless, it does not rotate within the shell and only moves translationally. Then, it should overall only contribute to the system's moment of inertia the same as a point mass.

The moment of inertia of the system about its center of mass is then $I_c = \frac{2}{3}MR^2$, so by the parallel axis theorem, the system's moment of inertia about an axis passing through the contact point is

$$I = I_c + 2MR^2 = \frac{8}{3}MR^2.$$

Also, the torque on the system about this axis is

$$\tau = |\vec{F} \times \vec{R}| = 2MgR\sin\theta.$$

Then, the angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{2MgR\sin\theta}{\frac{8}{3}MR^2} = \frac{3}{4} \cdot \frac{g\sin\theta}{R}.$$

Again, since the system rolls without slipping, we have that $a = R\alpha = \frac{3}{4}g\sin\theta$, so the answer is B.

Note: Try to derive the same result by starting from conservation of energy: $2Mgh = E_k^{(t)} + E_k^{(r)}$.

2013.13. If the rings were part of Saturn itself, they would have the same angular velocity ω as the rest of Saturn, so

$$v = R\omega \propto R.$$

Otherwise, if the rings were satellites, we know by the orbital speed formula that

$$\frac{mv^2}{R} = \frac{GMm}{R^2} \quad \Rightarrow \quad v_{orbit}^2 = \frac{GM}{R} \propto \frac{1}{R}.$$

Thus, the answer is A.

2013.14. We present two approaches to this problem.

Approach 1: Consider the center of mass frame. In this frame, the carts start with equal and opposite momenta, and the total momentum is zero. After the collision, the carts simply reverse the direction of their velocities (you can verify that this conserves energy and momentum). The difference between the cart of mass m's initial and final velocities is $12\,\text{m/s} - (-6\,\text{m/s}) = 18\,\text{m/s}$. In the center-of-mass frame, this cart therefore has initial velocity $18/2 = 9.0$ m/s and final velocity $-18/2 = -9.0$ m/s.

Thus, in the original frame of the problem, the velocity of the center of mass is 12 m/s – 9.0 m/s = 3.0 m/s, so the answer is B.

Approach 2: Let $v_1 = 12$ m/s and $v_2 = 0$ be the initial velocities of the carts, and let $w_1 = -6$ m/s and w_2 be the final velocities of the carts. Also, let $m_1 = m$ and $m_2 = 4.0$ kg. The elastic collision formula on the first cart yields

$$w_1 = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} = v_1\frac{m_1 - m_2}{m_1 + m_2}.$$

We can then solve for m_1 to get

$$\frac{m_1 - m_2}{m_1 + m_2} = \frac{w_1}{v_1} = \frac{-6\text{ m/s}}{12\text{ m/s}} = -\frac{1}{2} \quad \Rightarrow \quad m_1 = \frac{m_2}{3}.$$

Then, using momenta, the center of mass velocity is

$$v_c = \frac{m_1w_1}{m_1 + m_2} = \frac{1}{4}w_1 = 3.0\text{ m/s} \quad \Rightarrow \quad \text{the answer is B.}$$

2013.15. Let M be the rod's mass, L be its length, θ be the angle it makes with the horizontal, and ℓ be the length of the portion above the water. Since the rod is in equilibrium, we will solve this problem by balancing torques. There are two forces that contribute to this: gravity and buoyancy.

Gravity has a magnitude of Mg and acts vertically downward, at the middle of the rod, so the torque from gravity has magnitude

$$\tau_g = Mg\cos\theta \cdot \frac{L}{2}.$$

Meanwhile, the force of buoyancy is equal in magnitude to the weight of displaced water W, by Archimedes' Principle. To find this, we can use the density ratio given in the problem. If ρ_w is the density of water and ρ_r is the density of the rod, we have that

$$W = \frac{\rho_w}{\rho_r}Mg\left(1 - \frac{\ell}{L}\right).$$

The buoyant force acts at the midpoint of the portion of the rod under the water, which is distance $\frac{L+\ell}{2}$ from the pivot. Then, the torque from buoyancy has magnitude

$$\tau_b = W\cos\theta \cdot \frac{L+\ell}{2} = \frac{\rho_w}{\rho_r} Mg\cos\theta \left(1 - \frac{\ell}{L}\right) \cdot \frac{L+\ell}{2}.$$

These torques must balance, so

$$\tau_g = \tau_b \quad \Rightarrow \quad \frac{L}{2} = \frac{\rho_w}{\rho_r}\left(1 - \frac{\ell}{L}\right)\frac{L+\ell}{2}.$$

Dividing by $L/2$ and rearranging yields

$$\frac{\rho_r}{\rho_w} = \left(1 - \frac{\ell}{L}\right)\left(1 + \frac{\ell}{L}\right) = 1 - \left(\frac{\ell}{L}\right)^2,$$

$$\frac{\ell}{L} = \sqrt{1 - \frac{\rho_r}{\rho_w}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}.$$

Thus, the answer is D.

2013.16. A quick inspection of the answer choices reveals that they all have different length dimensions, so this problem can be solved with dimensional analysis. The dimensions of r are $[\,r\,] =$ m, and the dimensions of ρ_0 are $[\rho_0] =$ kg/m^3. With Newton's law of gravity, the dimensions of G are

$$F = \frac{Gm_1m_2}{r^2} \quad \Rightarrow \quad [G] = \left[F\frac{r^2}{m_1m_2}\right] = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}\cdot\frac{\text{m}^2}{\text{kg}^2} = \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}.$$

Then, the dimensions of $\frac{1}{\sqrt{G\rho_0}}$ are

$$\left[\frac{1}{\sqrt{G\rho_0}}\right] = \left[\left(\sqrt{G\rho_0}\right)^{-1/2}\right] = \left(\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\cdot\frac{\text{kg}}{\text{m}^3}\right)^{-1/2} = \text{s}.$$

Since all of the answer choices are of the form $0.5427\frac{r_0^\alpha}{\sqrt{G\rho_0}}$ for some power α, it follows that $\alpha = 0$ in order for the dimensions to match time. Thus, the answer is D.

2013.17. Let m be the magnitude of a single mass and ω be the angular velocity of the rod. Also, let $R \pm r$ be the radii of the two masses' orbits. Consider the rotating reference frame of the rod (referred to as the "co-rotating frame" in the problem statement). In this frame, there exists a centrifugal force of magnitude $m\omega(R \pm r)^2$ pushing outward on either mass. Since the outside mass has both a larger centrifugal force pushing it away from the center and a smaller force of gravity pulling it toward the center, there must be a force of tension T from the rod that pulls the outer mass toward the center.

This observation leaves us with answer choices B and D, which requires us to determine the stability of the equilibrium. Consider a small perturbation of the angle of the rod, as described in the problem statement. In the rotating reference frame, since the rod is in tension, the sum of the centrifugal force and gravity $F_c + F_g$ acts in a radial direction away from the rod for both masses. This leads to a restoring torque, rotating the rod back to its original radial orientation, so the equilibrium is stable. Thus, the answer is B.

2013.18. First, since the system is closed, conservation of momentum holds. The original net momentum of the system is zero, so the final momentum should be zero. This rules out answer choice B, which has momentum in the negative-x direction.

Next, since the momentum is zero, the velocity of the center of mass is also zero, which means that the center of mass should remain the same, at $(1, 0)$. This rules out answer choice A, which has a center of mass at $(0, 0)$.

Finally, an absence of external forces means that angular momentum is also conserved. In particular, since the linear momentum is zero, the angular momentum will be the same no matter which point we take to be the center of rotation[3]. The total angular momentum is then 2 kg m/s counterclockwise. This rules out C and D, which have angular momenta $\sqrt{2}$ kg m/s and 4 kg m/s counterclockwise, respectively, so the answer is E.

2013.19. This is a standard pendulum problem. Consider the mass at an angle θ, with velocity v, and let the tension in the rod be T. Draw a free body diagram and decompose the force of gravity mg into radial and tangential components. Considering the radial components of force, we have

$$F_{net} = ma_c = T - mg\cos\theta,$$

$$T = ma_c + mg\cos\theta = \frac{mv^2}{L} + mg\cos\theta.$$

Since both v^2 and $\cos\theta$ are maximized at the bottom of the swing, where $\theta = 0$ and potential energy is at its lowest, T is maximized here as well. Thus, the answer is B.

[3]You can derive this property based on the fact that the cross product distributes over vector addition. Also, had the linear momentum not been zero, we would need to be careful to select a fixed center about which to compute angular momenta.

2013.20. Use the notation from the previous problem. At the top of the swing, the mass has height $-L\cos\theta_0$, and at the bottom of the swing, it has height $-L$. Then, by conservation of energy, the kinetic energy of the mass at the bottom of the swing is

$$K = -\Delta U = -mg\Delta h = mgL(1-\cos\theta_0).$$

However, since $2K = mv^2$, we can plug this back into the formula for tension we previously derived to get, when $\theta = 0$:

$$T_{max} = mg\cos\theta + \frac{mv^2}{L} = mg + \frac{2K}{L} = mg(3-2\cos\theta_0).$$

Thus, the answer is E.

2013.21. Recall that when $\theta_0 \ll 1$, the period of a simple pendulum of length ℓ is well approximated by

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

However, when θ_0 is not $\ll 1$, this approximation is not good any more. This makes it difficult to distinguish between the answer choices. We can use a trick, though, to find the structure of the analogous formula for large values of θ_0. Notice that the period can only depend on ℓ, g, and θ_0, and furthermore, θ_0 is dimensionless. Then, by dimensional analysis, the period must be of the form[4]

$$T(\theta_0) = f(\theta_0)\sqrt{\frac{\ell}{g}}$$

for some function f. Since θ_0 and g remain constant in this problem, this just means that $T \propto \sqrt{\ell}$, so the answer is A.

2013.22. Since the entire weight of the student's body is supported by his toe, the normal force on the toe is equal to mg. Then, let $x_1 = 5$ cm be the distance from the Achilles tendon to the ankle joint, and let $x_2 = 20$ cm be the distance from the ankle joint to the toe. The net torque on the student's foot taken at the ankle joint is then equal to

$$\tau = Tx_1 - mgx_2.$$

Since the system is in equilibrium, this is zero, so

$$Tx_1 = mgx_2 \quad \Rightarrow \quad T = mg\cdot\frac{x_2}{x_1} = (60\,\text{kg})(10\ \text{m/s}^2)\cdot\frac{20\ \text{cm}}{5\ \text{cm}} = 2400\ \text{N}.$$

Thus, the answer is D.

[4]For an explicit power series for the function f, see the solution to Problem 2011.10

2013.23. Note that in this problem, $L = H$. We can obtain two equations from the information in the problem statement. First, when the man is at rest, all of his initial potential energy was converted to elastic potential energy in the cord, so

$$mgH = \frac{1}{2}kh^2.$$

Also, since the maximum tension is four times the man's weight,

$$4mg = kh.$$

Dividing the first equation by the second yields

$$\frac{\cancel{mg}H}{4\cancel{mg}} = \frac{\cancel{kh} \cdot h}{2\cancel{kh}} \quad \Rightarrow \quad h = \frac{H}{2}.$$

From here, there are several possible ways to write a formula for k, and the problem becomes a game of finding which answer choice matches up. After some experimenting, the second of our equations can be written as

$$k = \frac{4mg}{h} = \frac{8mg}{H}.$$

Thus, the answer is E.

2013.24. From the calculations in the previous problem, the answer is A.

2013.25. Let $\theta = 30°$. Since the block originally slides down the incline at a constant speed, we can balance forces to get

$$W \sin\theta - \mu W \cos\theta = 0 \quad \Rightarrow \quad \mu = \tan\theta.$$

When the horizontal force P is applied, we can decompose it into a parallel component $P \sin\theta$ and a perpendicular component $P \cos\theta$. The normal force balances out the combined perpendicular components of gravity and P, so

$$N = W \cos\theta + P \sin\theta.$$

Then, the force of kinetic friction on the block has magnitude

$$F_k = \mu N = \mu(W \cos\theta + P \sin\theta).$$

Now we can turn our attention to the parallel components. These must balance for the block to remain moving at a constant speed, so

$$P \cos\theta - F_k - W \sin\theta = 0 \quad \Rightarrow \quad P \cos\theta = \mu(W \cos\theta + P \sin\theta) + W \sin\theta.$$

From this point, we have at least two possible approaches:

Approach 1: Since we have $\sin\theta$ and $\cos\theta$ in this equation, we can use the trick of dividing both sides by $\cos\theta$. After this, recall that $\mu = \tan\theta$, so this simplifies to

$$P = \mu(W + \mu P) + \mu W.$$

Finally, we solve for P:

$$(1-\mu^2)P = 2\mu W \quad \Rightarrow \quad P = \frac{2\mu}{1-\mu^2}W.$$

Plugging in $\mu = \tan\theta = \frac{1}{\sqrt{3}}$, we are left with

$$P = \frac{\frac{2\sqrt{3}}{3}}{1-\frac{1}{3}}W = \sqrt{3}W \quad \Rightarrow \quad \text{the answer is D.}$$

Approach 2: We can express P from the previous equation and use the fact that $\mu = \tan\theta$ as follows

$$\begin{aligned} P &= W\,\frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta} \\ &= W\,\frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\ &= W\,\frac{\sin 2\theta}{\cos 2\theta} \\ &= W\tan 2\theta \;=\; W\tan 60^o \;=\; \sqrt{3}W \quad \Rightarrow \quad \text{the answer is D.} \end{aligned}$$

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Year 2014

$\boldsymbol{F = ma}$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2014

2014 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers will result in a deduction of $\frac{1}{4}$ point. There is no penalty for leaving an answer blank.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2014.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

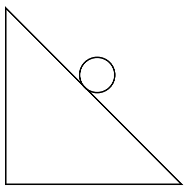
DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Contributors to this year's exam include David Fallest, David Jones, Jiajia Dong, Paul Stanley, Warren Turner, Qiuzi Li, and former US Team members Andrew Lin, Matthew Huang, Samuel Zbarsky.

1. A car turning to the right is traveling at constant speed in a circle. From the driver's perspective, the angular momentum vector about the center of the circle points X and the acceleration vector of the car points Y where

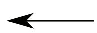
   (A) X is left, Y is left.
   (B) X is forward, Y is right.
   (C) X is down, Y is forward.
   (D) X is left, Y is right.
   (E) X is down, Y is right.

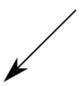
2. A ball rolls without slipping down an inclined plane as shown in the diagram.



Which of the following vectors best represents the direction of the total force that the ball exerts on the plane?

(A) 

(B) 

(C) 

(D) 

(E) 

3. An object of uniform density floats partially submerged so that 20% of the object is above the water. A 3 N force presses down on the top of the object so that the object becomes fully submerged. What is the volume of the object? The density of water is $\rho_{\mathrm{H_2O}} = 1000\ \mathrm{kg/m^3}$.

   (A) $V_{object} = 0.3$ L
   (B) $V_{object} = 0.67$ L
   (C) $V_{object} = 1.2$ L
   (D) $V_{object} = 1.5$ L
   (E) $V_{object} = 3.0$ L

4. What are the correct values of the numbers in the following statements? Assume there are no external forces, and take $N = 1$ to mean that the statement cannot be made for any meaningful number of particles.

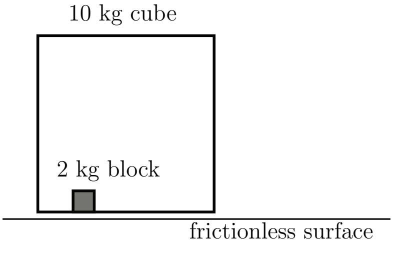
   - If a particle at rest explodes into N_1 or fewer particles with known masses, and the total kinetic energy of the new particles is known, the kinetic energy of each of the new particles is completely determined.
   - If a particle at rest explodes into N_2 or fewer particles, the velocities of the new particles must lie in a line.
   - If a particle at rest explodes into N_3 or fewer particles, the velocities of the new particles must lie in a plane.

   (A) $N_1 = 2,\ N_2 = 1,\ N_3 = 1$
   (B) $N_1 = 1,\ N_2 = 2,\ N_3 = 3$
   (C) $N_1 = 2,\ N_2 = 2,\ N_3 = 3$
   (D) $N_1 = 3,\ N_2 = 2,\ N_3 = 3$
   (E) $N_1 = 2,\ N_2 = 3,\ N_3 = 4$

5. A unicyclist goes around a circular track of radius 30 m at a (*amazingly fast!*) constant speed of 10 m/s. At what angle to the left (or right) of vertical must the unicyclist lean to avoid falling? Assume that the height of the unicyclist is much smaller than the radius of the track.

   (A) 9.46°
   (B) 9.59°
   (C) 18.4°
   (D) 19.5°
   (E) 70.5°

6. A cubical box of mass 10 kg with edge length 5 m is free to move on a frictionless horizontal surface. Inside is a small block of mass 2 kg, which moves without friction inside the box. At time $t = 0$, the block is moving with velocity 5 m/s directly towards one of the faces of the box, while the box is initially at rest. The coefficient of restitution for any collision between the block and box is 90%, meaning that the relative speed between the box and block immediately after a collision is 90% of the relative speed between the box and block immediately before the collision.



After 1 minute, the block is a displacement x from the original position. Which of the following is closest to x?

(A) 0 m
(B) 50 m
(C) 100 m
(D) 200 m
(E) 300 m

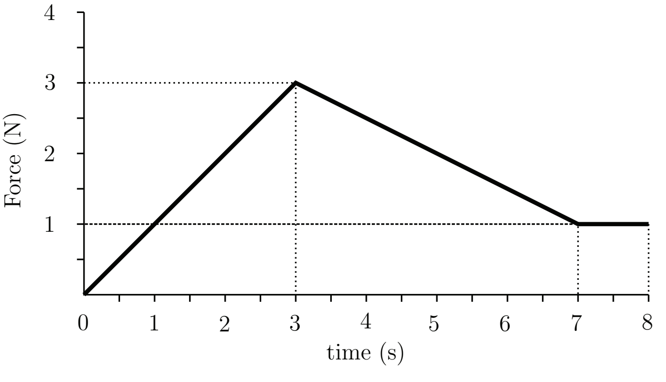
7. A 1.00 m long stick with uniform density is allowed to rotate about a point 30.0 cm from its end. The stick is perfectly balanced when a 50.0 g mass is placed on the stick 20.0 cm from the same end. What is the mass of the stick?

(A) 35.7 g
(B) 33.3 g
(C) 25.0 g
(D) 17.5 g
(E) 14.3 g

8. An object of mass M is hung on a vertical spring of spring constant k and is set into vertical oscillations. The period of this oscillation is T_0. The spring is then cut in half and the same mass is attached and the system is set up to oscillate on a frictionless inclined plane making an angle θ to the horizontal. Determine the period of the oscillations on the inclined plane in terms of T_0.

   (A) T_0
   (B) $T_0/2$
   (C) $2T_0 \sin\theta$
   (D) $T_0/\sqrt{2}$
   (E) $T_0 \sin\theta/\sqrt{2}$

9. A 5.0 kg object undergoes a time-varying force as shown in the graph below. If the velocity at $t = 0.0$ s is $+1.0$ m/s, what is the velocity of the object at $t = 7$ s?



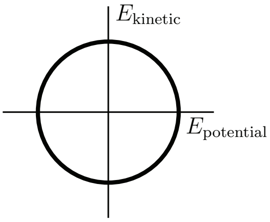
   (A) 2.45 m/s
   (B) 2.50 m/s
   (C) 3.50 m/s
   (D) 12.5 m/s
   (E) 15.0 m/s

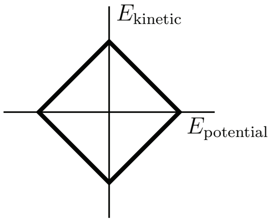
10. A radio controlled car is attached to a stake in the ground by a 3.00 m long piece of string, and is forced to move in a circular path. The car has an initial angular velocity of 1.00 rad/s and smoothly accelerates at a rate of 4.00 rad/s^2. The string will break if the centripetal acceleration exceeds 2.43×10^2 m/s^2. How long can the car accelerate at this rate before the string breaks?

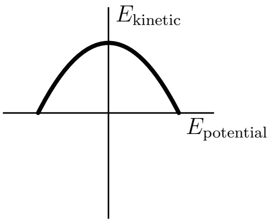
    (A) 0.25 s
    (B) 0.50 s
    (C) 1.00 s
    (D) 1.50 s
    (E) 2.00 s

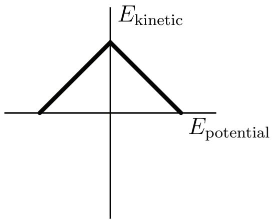
11. A point mass m is connected to an ideal spring on a horizontal frictionless surface. The mass is pulled a short distance and then released.

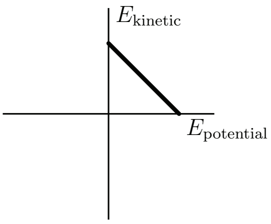
    Which of the following is the most correct plot of the kinetic energy as a function of potential energy?

(A) 

(B) 

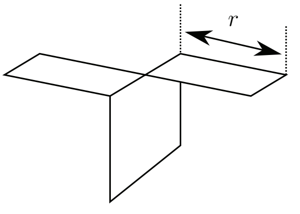
(C) 

(D) 

(E) 

The following information applies to questions 12 and 13

A paper helicopter with rotor radius r and weight W is dropped from a height h in air with a density of ρ.



Assuming that the helicopter quickly reaches terminal velocity, a function for the time of flight T can be found in the form

$$T = kh^{\alpha}r^{\beta}\rho^{\delta}W^{\omega}.$$

where k is an unknown dimensionless constant (actually, 1.164). α, β, δ, and ω are constant exponents to be determined.

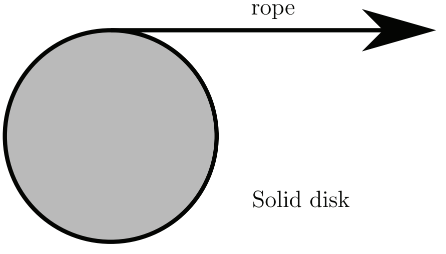
12. Determine α.

    (A) $\alpha = -1$
    (B) $\alpha = -1/2$
    (C) $\alpha = 0$
    (D) $\alpha = 1/2$
    (E) $\alpha = 1$

13. Determine β.

    (A) $\beta = 1/3$
    (B) $\beta = 1/2$
    (C) $\beta = 2/3$
    (D) $\beta = 1$
    (E) β can not be uniquely determined without more information.

14. A disk of moment of inertia I, mass M, and radius R has a cord wrapped around it tightly as shown in the diagram. The disk is free to slide on its side as shown in the top down view. A constant force of T is applied to the end of the cord and accelerates the disk along a frictionless surface.



After the disk has accelerated some distance, determine the ratio of the translational KE to total KE of the disk,

$$KE_{\text{translational}}/KE_{\text{total}} =$$

(A) $\frac{I}{MR^2}$
(B) $\frac{MR^2}{I}$
(C) $\frac{I}{3MR^2}$
(D) $\frac{I}{MR^2+I}$
(E) $\frac{MR^2}{MR^2+I}$

15. The maximum torque output from the engine of a new experimental car of mass m is τ. The maximum rotational speed of the engine is ω. The engine is designed to provide a constant power output P. The engine is connected to the wheels via a perfect transmission that can smoothly trade torque for speed with no power loss. The wheels have a radius R, and the coefficient of static friction between the wheels and the road is μ.

What is the maximum sustained speed v the car can drive up a 30 degree incline? Assume no frictional losses and assume μ is large enough so that the tires do not slip.

(A) $v = 2P/(mg)$
(B) $v = 2P/(\sqrt{3}mg)$
(C) $v = 2P/(\mu mg)$
(D) $v = \tau\omega/(mg)$
(E) $v = \tau\omega/(\mu mg)$

16. An object of mass m_1 initially moving at speed v_0 collides with an object of mass $m_2 = \alpha m_1$, where $\alpha < 1$, that is initially at rest. The collision could be completely elastic, completely inelastic, or partially inelastic. After the collision the two objects move at speeds v_1 and v_2. Assume that the collision is one dimensional, and that object one cannot pass through object two.

    After the collision, the speed ratio $r_1 = v_1/v_0$ of object 1 is bounded by

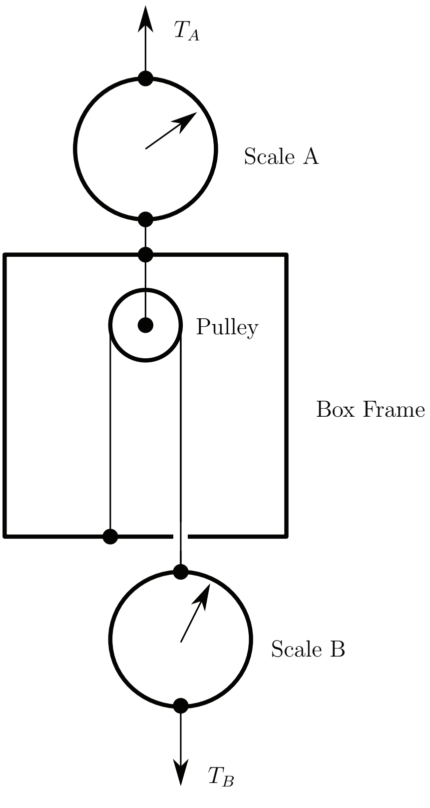
    (A) $(1-\alpha)/(1+\alpha) \leq r_1 \leq 1$
    (B) $(1-\alpha)/(1+\alpha) \leq r_1 \leq 1/(1+\alpha)$
    (C) $\alpha/(1+\alpha) \leq r_1 \leq 1$
    (D) $0 \leq r_1 \leq 2\alpha/(1+\alpha)$
    (E) $1/(1+\alpha) \leq r_1 \leq 2/(1+\alpha)$

17. A spherical cloud of dust in space has a uniform density ρ_0 and a radius R_0. The gravitational acceleration of free fall at the surface of the cloud due to the mass of the cloud is g_0.

    A process occurs (heat expansion) that causes the cloud to suddenly grow to a radius $2R_0$, while maintaining a uniform (but not constant) density. The gravitational acceleration of free fall at a point R_0 away from the center of the cloud due to the mass of the cloud is now

    (A) $g_0/32$
    (B) $g_0/16$
    (C) $g_0/8$
    (D) $g_0/4$
    (E) $g_0/2$

18. Consider the following diagram of a box and two weight scales. Scale A supports the box via a massless rope. A pulley is attached to the top of the box; a second massless rope passes over the pulley, one end is attached to the box and the other end to scale B. The two scales read indicate the tensions T_A and T_B in the ropes. Originally scale A reads 30 Newtons and scale B reads 20 Newtons.



If an additional force pulls down on scale B so that the reading increases to 30 Newtons, what will be the new reading on scale A?

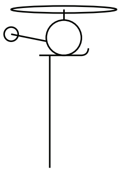
(A) 35 Newtons
(B) 40 Newtons
(C) 45 Newtons
(D) 50 Newtons
(E) 60 Newtons

Adapted from a demonstration by Richard Berg.

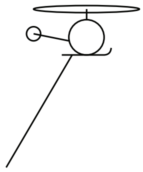
19. A helicopter is flying horizontally at constant speed. A perfectly flexible uniform cable is suspended beneath the helicopter; air friction on the cable is *not* negligible.

    Which of the following diagrams best shows the shape of the cable as the helicopter flies through the air to the right?

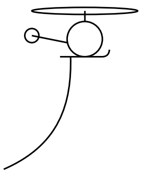
(A)



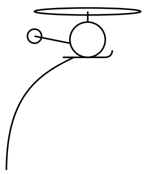
(B)



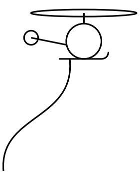
(C)



(D)



(E)

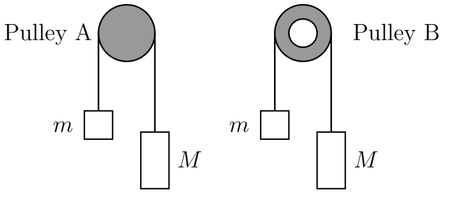


20. A crew of scientists has built a new space station. The space station is shaped like a wheel of radius R, with essentially all its mass M at the rim. When the crew arrives, the station will be set rotating at a rate that causes an object at the rim to have radial acceleration g, thereby simulating Earth's surface gravity. This is accomplished by two small rockets, each with thrust T newtons, mounted on the station's rim. How long a time t does one need to fire the rockets to achieve the desired condition?

    (A) $t = \sqrt{gR^3}\, M/(2T)$
    (B) $t = \sqrt{gR}\, M/(2T)$
    (C) $t = \sqrt{gR}\, M/T$
    (D) $t = \sqrt{gR/\pi}\, M/T$
    (E) $t = \sqrt{gR}\, M/(\pi T)$

    Adapted from a problem in *Physics for Scientists and Engineers* by Richard Wolfson

21. Two pulleys (shown in figure) are made of the same metal with density ρ. Pulley A is a uniform disk with radius R. Pulley B is identical except a circle of $R/2$ is removed from the center. When two boxes $M = \alpha m$ ($\alpha > 1$) are connected over the pulleys through a massless rope and move without slipping, what is the ratio between the accelerations in system A and B? The mass of pulley A is $M + m$.



(A) $a_A/a_B = 47/48$
(B) $a_A/a_B = 31/32$
(C) $a_A/a_B = 15/16$
(D) $a_A/a_B = 9/16$
(E) $a_A/a_B = 3/4$

22. A body of mass M and a body of mass $m \ll M$ are in circular orbits about their center of mass under the influence of their mutual gravitational attraction to each other. The distance between the bodies is R, which is much larger than the size of either body.

A small amount of matter $\delta m \ll m$ is removed from the body of mass m and transferred to the body of mass M. The transfer is done in such a way so that the orbits of the two bodies remain circular, and remain separated by a distance R. Which of the following statements is correct?

(A) The gravitational force between the two bodies increases.
(B) The gravitational force between the two bodies remains constant.
(C) The total angular momentum of the system increases.
(D) The total angular momentum of the system remains constant.
(E) The period of the orbit of two bodies remains constant.

The following information applies to questions 23 and 24

A 100 kg astronaut carries a launcher loaded with a 10 kg bowling ball; the launcher and the astronaut's spacesuit have negligible mass. The astronaut discovers that firing the launcher results in the ball moving away from her at a relative speed of 50 m/s.

23. What is the impulse delivered to the astronaut when firing the launcher?

    (A) 455 N s
    (B) 500 N s
    (C) 550 N s
    (D) 5000 N s
    (E) 5500 N s

24. The astronaut in the previous situation is now moving at 10 m/s (as measured in a certain frame of reference). She wishes to fire the launcher so that her velocity turns through as large an angle as possible (in this frame of reference). What is this maximum angle? (*Hint*: a diagram may be useful.)

    (A) 24.4°
    (B) 26.6°
    (C) 27.0°
    (D) 30.0°
    (E) 180.0°

25. A block with mass m is released from rest at the top of a frictionless ramp. The block starts at a height h_1 above the base of the ramp, slides down the ramp, and then up a second ramp. The coefficient of kinetic friction between the block and the second ramp is μ_k. If both ramps make an angle of θ with the horizontal, to what height h_2 above the base of the second ramp will the block rise?

    (A) $h_2 = (h_1 \sin\theta)/(\mu_k \cos\theta + \sin\theta)$
    (B) $h_2 = (h_1 \sin\theta)/(\mu_k + \sin\theta)$
    (C) $h_2 = (h_1 \sin\theta)/(\mu_k \cos^2\theta + \sin\theta)$
    (D) $h_2 = (h_1 \sin\theta)/(\mu_k \cos^2\theta + \sin^2\theta)$
    (E) $h_2 = (h_1 \cos\theta)/(\mu_k \sin\theta + \cos\theta)$

Answers, Problem Difficulty, and Topics

ToC

2014.1	E	⋆	circular motion, angular momentum
2014.2	E	⋆	forces, static friction
2014.3	D	⋆	Archimedes' Principle
2014.4	C	⋆ ⋆	conservation of linear momentum
2014.5	C	⋆ ⋆	forces, static friction
2014.6	B	⋆	center of mass, conservation of linear momentum
2014.7	D	⋆	torque
2014.8	D	⋆	springs, oscillations
2014.9	C	⋆	linear momentum
2014.10	E	⋆ ⋆	circular motion
2014.11	E	⋆	conservation of energy
2014.12	E	⋆	dimensional analysis
2014.13	D	⋆ ⋆	dimensional analysis
2014.14	D	⋆ ⋆ ⋆	kinetic energy
2014.15	A	⋆ ⋆	power
2014.16	B	⋆ ⋆ ⋆	collisions
2014.17	C	⋆ ⋆ ⋆	gravitation
2014.18	B	⋆	forces
2014.19	B	⋆ ⋆ ⋆	forces, tension, air friction
2014.20	B	⋆ ⋆	centripetal force, moment of inertia
2014.21	A	⋆ ⋆ ⋆	torque, generalized Atwood machine
2014.22	E	⋆ ⋆ ⋆	gravitation, binary stars
2014.23	A	⋆ ⋆	conservation of linear momentum
2014.24	C	⋆ ⋆ ⋆	conservation of linear momentum
2014.25	A	⋆ ⋆	kinetic friction, work

Solutions

ToC

2014.1. Let us talk about the easier vector first – the acceleration vector. During rotation with constant speed, the acceleration vector only has the centripetal (radial) component. Its magnitude is $a_c = \frac{v^2}{r} = \omega^2 r$, and it points in the direction of the center of rotation, in this case to the right. Therefore, Y is "right".

For the angular momentum vector, a bit of explanation will be appreciated by many students. The angular momentum vector of a point mass[1] is

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v}) = m(\vec{r} \times \vec{v}),$$

where the cross symbol denotes the vector product (this name reminds us that its result is a vector; it is also known as the cross product). In general, if $\vec{a} = \vec{b} \times \vec{c}$, then $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and its magnitude is $a = bc \sin\theta$, where θ is the angle between $\vec{b}$ and $\vec{c}$.

The vector product is to be distinguished from the scalar product (the result of which is a scalar; it is also known as the dot product and is used, for example, in the definition of work, $W = \vec{F} \cdot \vec{s} = Fs\cos\alpha$, where α is the angle between $\vec{F}$ and $\vec{s}$).

The direction of the vector product is determined using the *right-hand rule*[2]. To apply it here, point your right index finger in the direction of the radius vector (away from the center of rotation, as in Figure 1) and your middle finger in the direction of velocity (you will need to twist your hand a bit). Your thumb will be pointing down, so X is "down".

Alternatively, the direction can be determined by curling the fingers of your right hand to follow the car's trajectory, in which case your thumb will point down. Either way, the answer is E.

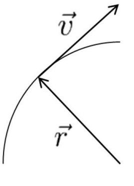


Figure 1: Illustration for Problem 2014.1.

[1]The car consists of infinitely many point masses, and all their angular moment are pointed in the same direction, so it suffices to consider just one of them.

[2]For more illustrations, see `https://en.wikipedia.org/wiki/Right-hand_rule`.

2014.2. The forces by which the ball acts on the wedge are illustrated in Figure 2. If the plane angle is θ, the normal force is $N = mg\cos\theta$ and acts perpendicular and into the plane. The static friction on the wedge is oriented down the plane and is equal to $F_s = \mu N = \mu mg\cos\theta$.

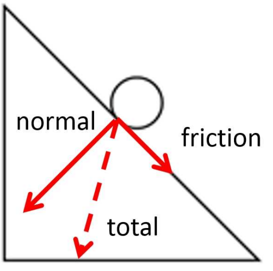


Figure 2: Illustration for Problem 2014.2.

Only answer E has both "into" and "down" components, so the answer is E.

2014.3. Let $y = 0.2$ be the fraction of the body that is not submerged, and so the fraction of the body that is submerged is $x = 1 - y = 0.8$. Then, from the Archimedes' Principle, we know that the force of buoyancy is equal to the weight of displaced water. Because the object is floating, it is also equal to the weight of the body, so

$$\rho_w gxV = \rho gV \quad \Rightarrow \quad \rho = x\rho_w \quad \text{(this turns out to be unimportant here).}$$

The additional force $F = 3\,\mathrm{N}$ submerges the body fully and so balances the new buoyancy

$$F = \rho_w gyV \quad \Rightarrow \quad V = \frac{F}{\rho_w gy} = 0.0015\ \mathrm{m}^3 = 1.5\ \mathrm{l} \quad \Rightarrow \quad \text{the answer is D.}$$

2014.4. Let us solve the easier parts of the problem first, N_2 and N_3:

- $N_2 = 2$, because
  - On one hand, it is possible for three particles to emerge with non-collinear velocities (consider the case where all of the particles have equal mass and speed, they emerge at the corners of an equilateral triangle, so their velocities are not collinear)
  - On the other hand, the total momentum of the daughter particles must be zero, therefore if there are two daughter particles, their velocities must be collinear (directed in opposite directions).

- $N_3 = 3$, because
  - On one hand, it is possible for four particles to emerge with non-coplanar velocities (consider the case where all of the particles have equal mass and speed, they emerge at the corners of a regular tetrahedron, so their velocities are not coplanar)
  - On the other hand, the total momentum of the daughter particles must be zero, therefore if there are three daughter particles, their velocities must be coplanar.
- $N_1 = 2$, because
  - If we consider three identical particles, any one of them could emerge with zero velocity. The total kinetic energy would be split between the other two particles, thus there are three possible solutions, i.e., the solution is not unique.
  - With two particles, we know that their velocities have to be collinear (pointing in opposite directions), and so we can write two equations for their speeds: conservation of linear momentum and conservation of kinetic energy. Once we determine their speeds, since their masses are known, their kinetic energies are uniquely determined.

With $(N_1, N_2, N_3) = (2, 2, 3)$, the answer is C.

2014.5. The forces acting on the unicycle are:

- The normal force from the ground reacting to the weight mg and acting upwards at the point of contact.
- The force of static friction, which is the centripetal force, acting at the point of contact and pointing horizontally, towards the center of the circle.

If the unicyclist balances the unicycle so that it leans in the direction of the sum of these two forces, there will be no torque on the system, allowing her to stay upright (this is an unstable equilibrium so, in reality, the unicyclist will have to actively balance the unicycle around that optimal angle). If θ is the angle that the sum of the two forces makes with the vertical, we can write

$$\tan\theta = \frac{F_s}{N} = \frac{\frac{mv^2}{r}}{mg} = \frac{v^2}{gr}.$$

Finally,

$$\theta = \arctan\frac{v^2}{gr} = 18.4^o \quad \Rightarrow \quad \text{the answer is C.}$$

2014.6. In principle, we could try to calculate the exact motion of the block, its collisions with the box walls, and the resulting motion of the box. However, it is much much simpler to consider the motion of the center of mass of the block-box system because, by conservation of linear momentum, the center of mass is moving at a constant speed throughout this experiment. The initial speed of the block is $v = 5$ m/s and the masses of of the box and the block are $M = 10$ kg and $m = 2$ kg, so the speed of the center of mass is

$$v_c = \frac{mv}{M+m} = \frac{5}{6} \text{ m/s}.$$

Over the duration of the experiment, $t = 60$ s, the center of mass has moved

$$s = v_c t = 50 \text{ m}.$$

Conservatively speaking, the displacement x of the block must be within the size of the box ($a = 5\,\text{m}$) from s, so $45 \text{ m} \le x \le 55$ m, and we see that the answer is B.

2014.7. Let $L = 1$ m, $x = 30$ cm, $y = 20$ cm, and $m = 50$ g. Let the unknown mass of the stick be M. The centroid of the stick is $L/2 - x$ from the center of rotation, while the additional mass m is at distance $x - y$ from the center of rotation. The condition for balance is

$$M\left(\frac{L}{2} - x\right) = m(x - y) \quad \Rightarrow \quad M = m\frac{x-y}{\frac{L}{2} - x} = 25 \text{ g},$$

so the answer is C.

2014.8. The force constant of a spring is inversely proportional to its rest length, so cutting a spring in half doubles the force constant of the two resulting parts. Placing the system on an incline changes the equilibrium position of the mass, but does not affect the period of oscillation. Therefore, the new period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{2k}} = \frac{1}{\sqrt{2}}\,2\pi\sqrt{\frac{m}{k}} = \frac{T_0}{\sqrt{2}},$$

so the answer is D.

2014.9. The area under the F vs. t graph is a measure of the change in momentum, so

$$\Delta p = \frac{1}{2}(3\,\text{N})(3\,\text{s}) + \frac{1}{2}(3\,\text{N} + 1\,\text{N})(7\,\text{s} - 3\,\text{s}) = 12.5\,\text{Ns},$$

therefore the additional speed is $\Delta v = \Delta p/m = 2.5$ m/s, and the final speed is $v = v_0 + \Delta v = 3.5$ m/s, so the answer is C.

2014.10. The centripetal acceleration is

$$a_c = r\omega^2,$$

while ω as a function of time is

$$\omega = \omega_0 + \frac{\alpha t^2}{2}.$$

The condition for the string to break is

$$r\left(\omega_0 + \frac{\alpha t^2}{2}\right)^2 = a_c^{max},$$

and solving for t, we get

$$t = \sqrt{\frac{2}{\alpha}\left(\sqrt{\frac{a_c^{max}}{r}} - \omega_0\right)} = 2 \text{ s}.$$

Therefore, the answer is E.

2014.11. In this case the mechanical energy is conserved, i.e.,

$$E_k + E_p = \text{const},$$

so the graph is described by

$$E_k = \text{const} - E_p,$$

and is given by a straight line with slope of -1. Therefore, the answer is E.

2014.12. When the helicopter is at terminal velocity (and the assumption is that its velocity converges quickly), it is falling at constant speed. As such, $h = vT$, where v is the terminal velocity. In that case, $\alpha = 1$ and the answer is E.

2014.13. Using the dimensional analysis, and knowing from the previous problem that $\alpha = 1$, we can write (recall that $\text{N} = \text{kg}\,\text{m}/\text{s}^2$)

$$T = kh^\alpha r^\beta \rho^\delta W^\omega \quad \Rightarrow \quad \text{s} = \text{m}^1\ \text{m}^\beta\ (\text{kg}/\text{m}^3)^\delta\ (\text{kg}\,\text{m}/\text{s}^2)^\omega.$$

Equating the powers of individual basic units on the left- and right-hand sides, we get

$$\begin{array}{lcl}
\text{for s} & \Rightarrow & \omega = -\dfrac{1}{2} \\[2ex]
\text{for kg} & \Rightarrow & \delta = -\omega = \dfrac{1}{2} \\[2ex]
\text{for m} & \Rightarrow & 0 = 1 + \beta - 3\delta + \omega \quad \Rightarrow \quad \beta = 1,
\end{array}$$

so the answer is D.

2014.14. When a force acts on a free (unattached) object exactly at its centroid, it will give it only translatory motion. When a force acts at a point other than the centroid, it will give it

- translatory motion as if it acts at the centroid, and
- rotational motion with angular acceleration $\alpha = \tau/I$, where τ is the torque due to the force and I is the moment of inertia of the object with respect to its centroid.

The work done by the force is converted partially to translational (linear) kinetic energy and the rest to rotational kinetic energy. In this case $\tau = TR$, so

$$E_k^{transl} = \frac{mv^2}{2} = \frac{m}{2}(at)^2 = \frac{m}{2}\left(\frac{Tt}{m}\right)^2 = \frac{T^2t^2}{2m},$$

and

$$E_k^{rot} = \frac{I\omega^2}{2} = \frac{I}{2}(\alpha t)^2 = \frac{I}{2}\left(\frac{\tau t}{I}\right)^2 = \frac{T^2R^2t^2}{2I}.$$

Finally,

$$\frac{E_k^{transl}}{E_k} = \frac{E_k^{transl}}{E_k^{transl} + E_k^{rot}} = \frac{\frac{T^2t^2}{2m}}{\frac{T^2t^2}{2m} + \frac{T^2R^2t^2}{2I}} = \frac{I}{I + MR^2},$$

so the answer is D.

2014.15. This problem may be confusing for many students, primarily because it gives way too much information. Here we use the formula that relates the power P, force F, and speed v:

$$P = Fv.$$

It is instructive to see how it can be derived:

$$P = \frac{\Delta W}{\Delta t} = \frac{F\Delta s}{\Delta t} = F\frac{\Delta s}{\Delta t} = Fv.$$

Since we are looking for the maximum *sustained* or *constant* speed, we know that the engine force is balanced by the component of gravity parallel to the incline:

$$F = mg\sin\theta,$$

and so

$$v = \frac{P}{F} = \frac{P}{mg\sin\theta} = \frac{2P}{mg}.$$

Therefore, the answer is A.

2014.16. This problem is almost identical to Problem 2016.13. Let us make a slight change in the notation for this problem in order to be able to explore what happens in a more general case:

- Let the velocities of the two objects before the collision be v_1 and v_2.
- Let the velocities of the two objects after the collision be w_1 and w_2.
- Then the ratio r_1 is defined as $r_1 = \left|\frac{w_1}{v_1}\right|$.

One extreme case of collisions are *elastic collisions* (think air hockey pucks), for which both linear momentum and kinetic energy are conserved, so

$$m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2 \quad \text{and} \quad \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} = \frac{m_1w_1^2}{2} + \frac{m_2w_2^2}{2}.$$

These equations imply $v_1 + w_1 = v_2 + w_2$ (a useful formula in its own right), and finally

$$w_1 = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \qquad \text{and} \qquad w_2 = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2}.$$

The other extreme case of collisions are *perfectly inelastic collisions* (think two ice cream scoops hitting each other and merging in the process). In that case, only the linear momentum is conserved,

$$m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2,$$

and objects get merged, so $w_1 = w_2 = w$. Solving for w, we get

$$w = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}.$$

Back to our problem, with $v_2 = 0$, we find that for elastic collisions

$$r_1 = \left|\frac{w_1}{v_1}\right| = \frac{m_1 - m_2}{m_1 + m_2} = \frac{1-\alpha}{1+\alpha},$$

where $\alpha < 1$, so there are no issues with direction reversals. For perfectly inelastic collisions we have

$$r_1 = \left|\frac{w_1}{v_1}\right| = \frac{m_1}{m_1 + m_2} = \frac{1}{1+\alpha}.$$

It can be shown that with respect to r_1, all other collisions (partially inelastic collisions) lie between these two extremes[3] and, in general,

$$\frac{1-\alpha}{1+\alpha} \le r_1 \le \frac{1}{1+\alpha}, \qquad \text{which is answer B.}$$

[3] In the center of mass frame, the final velocity is monotonic with the kinetic energy, because the velocities of the two masses are proportional. To transform back into the original frame, we add v_{cm} back. This means that the signed final velocity is indeed bounded by the extreme cases (completely elastic and completely inelastic collisions).

2014.17. Using the tools of integration (which we don't have to know here, but there is no reason not to utilize the results derived using them), Newton showed that

- The force of gravity of a spherically symmetric body on an external object is the same as if all of the body's mass was concentrated at its center.
- If the body is a spherically symmetric shell or a hollow ball, objects anywhere inside of it do not "feel" its gravity.

This implies that, for example, if we are deep in a mine on a perfectly spherically symmetric planet, we only feel the gravity due to the parts of the planet that are closer to its center than we are.

In the case of the dust cloud, the gravity due to the portion of the cloud beyond the radius where we are is 0. What is left within the original sphere's radius? Since the density changes but remains uniform, the mass within the original radius is only 1/8 of the original mass, therefore the answer is $g = g_0/8$ and the result is C.

2014.18. Regardless of what is in the box, the only thing that matters here is that we pull down on scale B with additional 10 N, so the force on scale A must increase by 10 N too. Therefore the result is 30 N + 10 N = 40 N and the result is B.

2014.19. Consider cable tension T at any point P of the cable. We can say that T comes from two forces exerted on the part of the cable below P:

- the weight of that part of the cable, which gives T its vertical component, and
- air friction on that part of the cable, which gives T its horizontal component.

As we look at different points P along the cable, we see that these two forces are always proportional to the length of the cable below P, and so T has the same direction at any point on the cable. For this reason, the cable will be shaped as a straight line, so the answer is B.

Note: If the cable did not produce any friction, the answer would be A. If there was a weight at the end of the cable producing negligible friction, the answer would be D. If there was a light object at the end of the cable producing significant friction (perhaps a kite), while the air friction on the cable was negligible, the answer would be C. Answer E does not make much sense in this context.

2014.20. For the new space station to simulate gravity, the centripetal acceleration at the rim must be g, so $\omega^2 R = g \;\Rightarrow\; \omega = \sqrt{g/R}$. All of the mass M is distributed on the perimeter of the station, so its moment of inertia is $I = MR^2$, and Euler's Second Law becomes $\tau = I\alpha = MR^2\alpha$. On the other hand, torque comes from the thrust of the two engines acting tangentially at distance R from the center, hence

$$\tau = 2TR \quad \Rightarrow \quad \alpha = \frac{2TR}{MR^2} = \frac{2T}{MR}.$$

The time needed is

$$t = \frac{\omega}{\alpha} = \frac{\sqrt{gR}M}{2T} \quad \Rightarrow \quad \text{the answer is B.}$$

2014.21. In this problem we are dealing with a generalized Atwood machine, so let us begin by determining the moments of inertia of the two pulleys. Pulley A is a solid cylinder, so

$$I_A = \frac{1}{2}(M+m)R^2.$$

Pulley B is a solid cylinder with a smaller solid cylinder removed. The smaller, missing cylinder had radius $R/2$, so it contained $1/4$ of the total mass of the full cylinder. Therefore,

$$I_B = \frac{1}{2}(M+m)R^2 - \frac{1}{2}\frac{M+m}{4}\left(\frac{R}{2}\right)^2 = \frac{15}{32}(M+m)R^2.$$

Note that for a generalized Atwood machine the rope tensions on two sides of the pulley are not equal. The following equations are valid for each pulley:

$$Ma = Mg - T_M \qquad ma = T_m - mg \qquad I\alpha = R(T_M - T_m).$$

Eliminating the two tensions T_M and T_m we get

$$a = \frac{(M-m)g}{I/R^2 + M + m}.$$

Now to the question:

$$\frac{a_A}{a_B} = \frac{\frac{(M-m)g}{I_A/R^2+M+m}}{\frac{(M-m)g}{I_B/R^2+M+m}} = \frac{\frac{I_B}{R^2}+M+m}{\frac{I_A}{R^2}+M+m} = \frac{\frac{I_B}{(M+m)R^2}+1}{\frac{I_A}{(M+m)R^2}+1} = \frac{\frac{15}{32}+1}{\frac{1}{2}+1} = \frac{47}{48},$$

so the answer is A.

2014.22. Let us derive the formula for the period of a binary system whose stars are on circular orbits. The center of mass is at distances $r_1 = \frac{mR}{M+m}$ and $r_2 = \frac{MR}{M+m}$ from M and m, respectively. This comes from $r_1 + r_2 = R$ and $Mr_1 = mr_2$. The centripetal force is their mutual gravity, so if their angular velocity is ω

$$Mr_1\omega^2 = G\frac{Mm}{R^2} \quad \Rightarrow \quad \omega = \sqrt{\frac{GMm}{Mr_1R}} = \sqrt{\frac{GMm}{M\frac{mR}{M+m}R^2}} = \sqrt{\frac{G(M+m)}{R^3}}.$$

Therefore,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{G(M+m)}}.$$

From this formula we see that if both R, the distance between stars, and $M+m$, their total mass, remained constant, as is the case in our problem, then their period of rotation remains constant too, and the answer is E.

Note: Here we show that in this mass transfer, both the gravity and the total angular momentum decrease, so answers A-D are incorrect.
Originally, the force of gravity is

$$F = G\,\frac{Mm}{R^2}.$$

If mass δm is transfered, the new force of gravity is

$$F' = G\,\frac{(M+\delta m)(m-\delta m)}{R^2} = G\,\frac{Mm-(M-m)\delta m-(\delta m)^2}{R^2}.$$

Therefore,

$$F' \approx G\,\frac{Mm-(M-m)\delta m}{R^2} \approx G\,\frac{Mm-\delta m}{R^2} = G\,\frac{Mm}{R^2}\left(1-\frac{\delta m}{m}\right).$$

Finally,

$$F' \;\approx\; \left(1-\frac{\delta m}{m}\right)F \;<\; F.$$

Total angular momentum is

$$L = L_M + L_m,$$

where

$$L_M = Mv_1r_1 = Mv_1\frac{mR}{M+m}. \tag{1}$$

Since gravity is the centripetal force, we can write

$$\frac{Mv_1^2}{r_1} = G\frac{Mm}{R^2} \quad \Rightarrow \quad v_1 = m\sqrt{\frac{G}{(M+m)R}}.$$

Combined with Equation 1, this yields

$$L_M = M^2 m\sqrt{\frac{GR}{(M+m)^3}}.$$

Similarly,

$$L_m = Mm^2\sqrt{\frac{GR}{(M+m)^3}},$$

so

$$L = L_M + L_m = Mm(M+m)\sqrt{\frac{GR}{(M+m)^3}} = Mm\sqrt{\frac{GR}{M+m}}.$$

During the transfer in this problem, R and $M+m$ remain constant. We already saw that the quantity Mm decreases as δm is transferred from m to M, therefore $L' < L$.

2014.23. Consider the astronaut-ball system. The trick here is to recognize that $v_r = 50$ m/s is not the velocity of the ball with respect to their center of mass but with respect to the astronaut, who also starts to move after the launch. Let us switch to the reference coordinate system of their center of mass. Let $m_1 = 100$ kg and $m_2 = 10$ kg and let $v_1 < 0$ and $v_2 > 0$ be the velocities of the astronaut and the ball, respectively. Then

$$v_r = v_2 - v_1,$$

and from the conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = 0.$$

Eliminating v_2 we get

$$m_1 v_1 + m_2(v_r + v_1) = 0 \quad \Rightarrow \quad v_1 = -\frac{m_2 v_r}{m_1 + m_2} \quad \Rightarrow \quad m_1 v_1 = -\frac{m_1 m_2 v_r}{m_1 + m_2}.$$

The impulse that was delivered to the astronaut by the launch recoil is

$$J = \Delta p_1 = m_1 v_1 - 0 = -\frac{m_1 m_2 v_r}{m_1 + m_2} = -454.55 \text{ N s}.$$

The desired quantity is $|J|$, so the best answer is A.

2014.24. The final velocity $\vec{v}_f$ of the astronaut is the vector sum of her initial velocity $\vec{v}_i$ and the additional velocity $\Delta\vec{v}$ due to the recoil impulse $\vec{J}$, where $\Delta\vec{v} = \vec{J}/m_1$ and the magnitude of $\vec{J}$ is the result of the previous problem. Figure 3 shows that $\vec{v}_f$ lies on a circle, and we see that the greatest deflection angle is achieved when the angle between $\vec{v}_f$ and $\Delta\vec{v}$ is right.

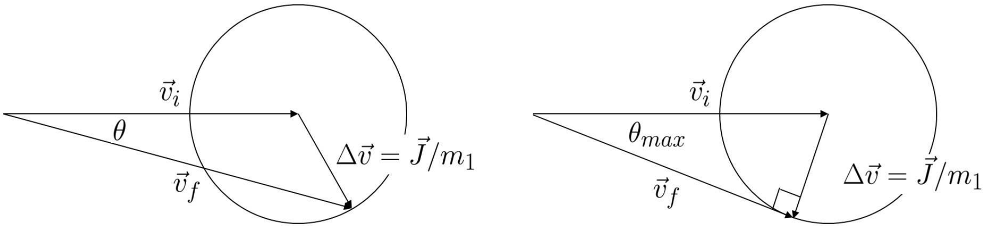


Figure 3: Illustration for Problem 2014.24.

Therefore,

$$\sin\theta_{max} = \frac{J}{m_1 v_i} = 0.4545 \qquad \Rightarrow \qquad \theta_{max} = 27^o,$$

so the answer is C.

2014.25. The initial potential energy mgh_1 is converted into the final potential energy mgh_2 and the work against kinetic friction

$$W_k = F_k s = (\mu_k mg\cos\theta)\left(\frac{h_2}{\sin\theta}\right) = \frac{\mu_k mgh_2\cos\theta}{\sin\theta},$$

so we can write

$$mgh_1 = mgh_2 + \frac{\mu_k mgh_2\cos\theta}{\sin\theta} \qquad \Rightarrow \qquad h_2 = \frac{h_1\sin\theta}{\sin\theta + \mu_k\cos\theta},$$

so the answer is A.

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Year 2015

$\boldsymbol{F = ma}$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2015

2015 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

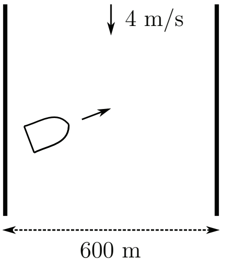
INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2015.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

1. A 600 meter wide river flows directly south at 4.0 m/s. A small motor boat travels at 5.0 m/s in still water and points in such a direction so that it will travel directly east relative to the land.



The time it takes to cross the river is closest to

(A) 67 s
(B) 120 s
(C) 150 s
(D) 200 s
(E) 600 s

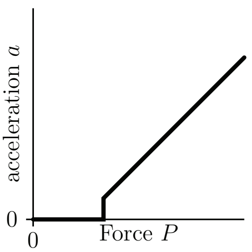
2. A car travels directly north on a straight highway at a constant speed of 80 km/hr for a distance of 25 km. The car then continues directly north at a constant speed of 50 km/hr for a distance of 75 more kilometers. The average speed of the car for the entire journey is closest to

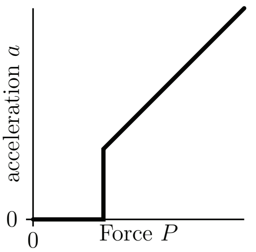
(A) 55.2 km/hr
(B) 57.5 km/hr
(C) 65 km/hr
(D) 69.6 km/hr
(E) 72.5 km/hr

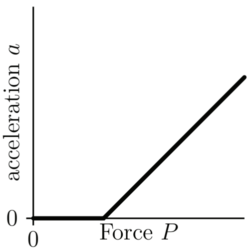
3. The force of friction on an airplane in level flight is given by $F_f = kv^2$, where k is some constant, and v is the speed of the airplane. When the power output from the engines is P_0, the plane is able to fly at a speed v_0. If the power output of the engines is increased by 100% to $2P_0$, the airplane will be able to fly at a new speed given by

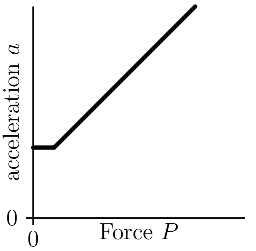
   (A) $1.12v_0$
   (B) $1.26v_0$
   (C) $1.41v_0$
   (D) $2.82v_0$
   (E) $8v_0$

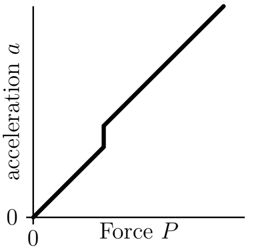
4. A 2.0 kg box is originally at rest on a horizontal surface where the coefficient of static friction between the box and the surface is μ_s and the coefficient of the kinetic friction between the box and the surface is $\mu_k = 0.90\mu_s$. An external horizontal force of magnitude P is then applied to the box. Which of the following is a graph of the acceleration of the box a versus the external force P?

(A)


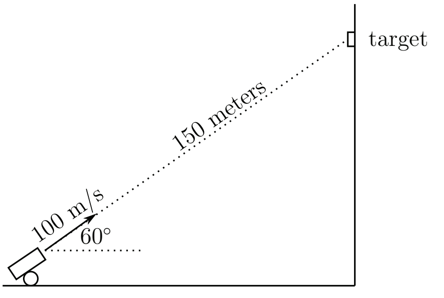
(B)


(C)


(D)


(E)


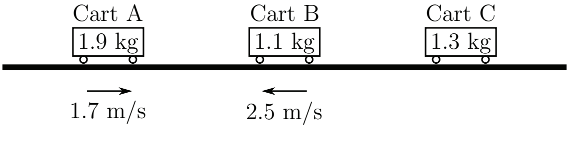
5. A 470 gram lead ball is launched at a 60 degree angle above the horizontal with an initial speed of 100 m/s directly toward a target on a vertical cliff wall that is 150 meters away as shown in the figure.



Ignoring air friction, by what distance does the lead ball miss the target when it hits the cliff wall?

(A) 1.3 m
(B) 2.2 m
(C) 5.0 m
(D) 7.1 m
(E) 11 m

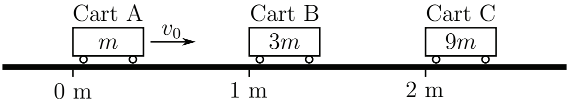
6. Three trolley carts are free to move on a one dimensional frictionless horizontal track. Cart A has a mass of 1.9 kg and an initial speed of 1.7 m/s to the right; Cart B has a mass of 1.1 kg and an initial speed of 2.5 m/s to the left; cart C has a mass of 1.3 kg and is originally at rest. Collisions between carts A and B are perfectly elastic; collisions between carts B and C are perfectly inelastic.



What is the velocity of the center of mass of the system of the three carts after the last collision?

(A) 0.11 m/s
(B) 0.16 m/s
(C) 1.4 m/s
(D) 2.0 m/s
(E) 3.23 m/s

The following information applies to questions 7 and 8



Carts A, B, and C are on a long horizontal frictionless track. The masses of the carts are m, $3m$, and $9m$. Originally cart B is at rest at the 1.0 meter mark and cart C is at rest on the 2.0 meter mark. Cart A is originally at the zero meter mark moving toward the cart B at a speed of v_0.

7. Assuming that all collisions are completely *inelastic*, what is the final speed of cart C?

   (A) $v_0/13$
   (B) $v_0/10$
   (C) $v_0/9$
   (D) $v_0/3$
   (E) $2v_0/5$

8. Assuming that all collisions are completely *elastic*, what is the final speed of cart C?

   (A) $v_0/8$
   (B) $v_0/4$
   (C) $v_0/2$
   (D) v_0
   (E) $2v_0$

The following information applies to questions 9 and 10

A 0.650 kg ball moving at 5.00 m/s collides with a 0.750 kg ball that is originally at rest. After the collision, the 0.750 kg ball moves off with a speed of 4.00 m/s, and the 0.650 kg ball moves off at a right angle to the final direction of motion of the 0.750 kg ball.

9. What is the final speed of the 0.650 kg ball?

   (A) 1.92 m/s
   (B) 2.32 m/s
   (C) 3.00 m/s
   (D) 4.64 m/s
   (E) 5.77 m/s

10. Let the change in total kinetic energy in this collision be defined by $\Delta K = K_f - K_i$, where K_f is the total final kinetic energy, and K_i is the total initial kinetic energy. Which of the following is true?

    (A) $\Delta K = (K_i + K_f)/2$
    (B) $K_f < \Delta K < K_i$
    (C) $0 < \Delta K < K_f$
    (D) $\Delta K = 0$
    (E) $-K_i < \Delta K < 0$

11. A sphere floats in water with 2/3 of the volume of the sphere submerged. The sphere is removed and placed in oil that has 3/4 the density of water. If it floats in the oil, what fraction of the sphere would be submerged in the oil?

    (A) 1/12
    (B) 1/2
    (C) 8/9
    (D) 17/12
    (E) The sphere will not float, it will sink in the oil.

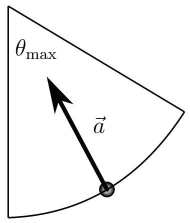
The following information applies to questions 12 and 13

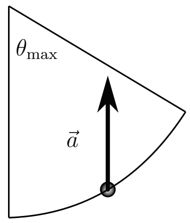
A pendulum consists of a small bob of mass m attached to a fixed point by a string of length L. The pendulum bob swings down from rest from an initial angle $\theta_{\text{max}} < 90$ degrees.

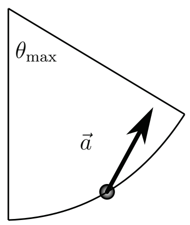
12. Which of the following statements about the pendulum bob's acceleration is true?

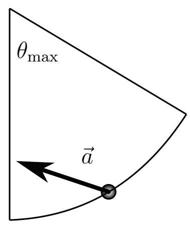
    (A) The magnitude of the acceleration is constant for the motion.
    (B) The magnitude of the acceleration at the lowest point is g, the acceleration of free fall.
    (C) The magnitude of the acceleration is zero at some point of the pendulum's swing.
    (D) The acceleration is always directed toward the center of the circle.
    (E) The acceleration at the bottom of the swing is pointing vertically upward.

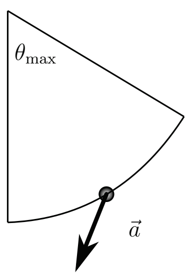
13. Consider the pendulum bob when it is at an angle $\theta = \frac{1}{2}\theta_{\text{max}}$ on the way up (moving toward θ_{max}). What is the direction of the acceleration vector?

(A) 

(B) 

(C) 

(D) 

(E) 

The following information applies to questions 14 and 15

A 3.0 meter long massless rod is free to rotate horizontally about its center. Two 5.0 kg point objects are originally located at the ends of the rod; they are free to slide on the frictionless rod and are kept from flying off the rod by an inflexible massless rope that connects the two objects.

Originally the system is rotating at 4.0 radians per second; assume the system is completely frictionless; and ignore any concerns about instability of the system.

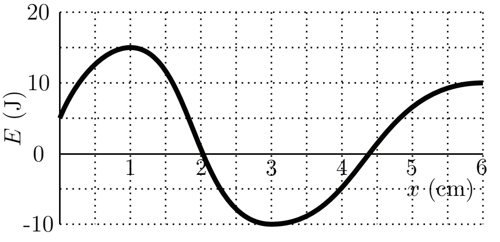
14. Calculate the original tension in the rope.

(A) 60 N
(B) 106 N
(C) 120 N
(D) 240 N
(E) 480 N

15. The rope is slowly tightened by a small massless motor attached to one of the objects. It is done in such a way as to pull the two objects closer to the center of the rotating rod. How much work is done by the motor in pulling the two objects from the ends of the rod until they are each 0.5 meters from the center of rotation?

    (A) 120 J
    (B) 180 J
    (C) 240 J
    (D) 1440 J
    (E) 1620 J

16. Shown below is a graph of potential energy as a function of position for a 0.50 kg object.



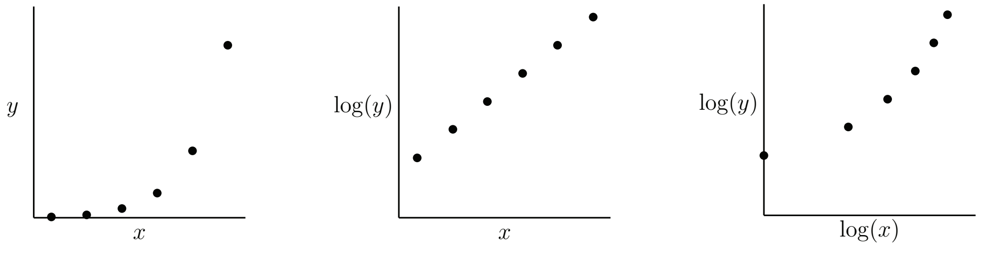
Which of the following statements is NOT true in the range 0 cm $< x <$ 6 cm?

    (A) The object could be at equilibrium at either $x = 1$ cm or $x = 3$ cm.
    (B) The minimum possible total energy for this object in the range is -10 J.
    (C) The magnitude of the force on the object at 4 cm is approximately 1000 N.
    (D) If the total energy of the particle is 0 J then the object will have a maximum kinetic energy of 10 J.
    (E) The magnitude of the acceleration of the object at $x = 2$ cm is approximately 4 cm/s^2.

17. A flywheel can rotate in order to store kinetic energy. The flywheel is a uniform disk made of a material with a density ρ and tensile strength σ (measured in Pascals), a radius r, and a thickness h. The flywheel is rotating at the maximum possible angular velocity so that it does not break. Which of the following expression correctly gives the maximum kinetic energy per kilogram that can be stored in the flywheel? Assume that α is a dimensionless constant.

    (A) $\alpha\sqrt{\rho\sigma/r}$
    (B) $\alpha h\sqrt{\rho\sigma/r}$
    (C) $\alpha\sqrt{(h/r)}(\sigma/\rho)^2$
    (D) $\alpha(h/r)(\sigma/\rho)$
    (E) $\alpha\sigma/\rho$

18. Shown below are three graphs of the same data.



Which is the correct functional relationship between the data points? Assume a and b are constants.

    (A) $y = ax + b$
    (B) $y = ax^2 + b$
    (C) $y = ax^b$
    (D) $y = ae^{bx}$
    (E) $y = a\log x + b$

The following information applies to questions 19 and 20

A U-tube manometer consists of a uniform diameter cylindrical tube that is bent into a U shape. It is originally filled with water that has a density ρ_w. The *total* length of the column of water is L. Ignore surface tension and viscosity.

19. The water is displaced slightly so that one side moves up a distance x and the other side lowers a distance x. Find the frequency of oscillation.

    (A) $\frac{1}{2\pi}\sqrt{2g/L}$
    (B) $2\pi\sqrt{g/L}$
    (C) $\frac{1}{2\pi}\sqrt{2L/g}$
    (D) $\frac{1}{2\pi}\sqrt{g/\rho_w}$
    (E) $2\pi\sqrt{\rho_w gL}$

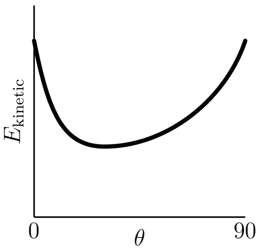
20. Oil with a density half that of water is added to one side of the tube until the total length of oil is equal to the total length of water. Determine the equilibrium height difference between the two sides

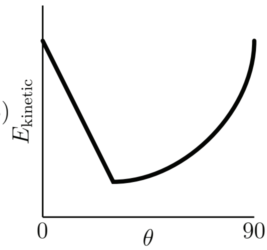
    (A) L
    (B) $L/2$
    (C) $L/3$
    (D) $3L/4$
    (E) $L/4$

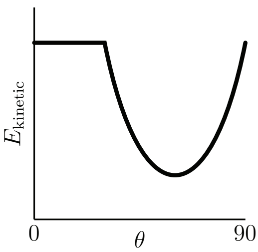
21. An object launched vertically upward from the ground with a speed of 50 m/s bounces off of the ground on the return trip with a coefficient of restitution given by $C_R = 0.9$, meaning that immediately after a bounce the upward speed is 90% of the previous downward speed. The ball continues to bounce like this; what is the total amount of time between when the ball is launched and when it finally comes to a rest? Assume the collision time is zero; the bounce is instantaneous. Treat the problem as ideally classical and ignore any quantum effects that might happen for very small bounces.

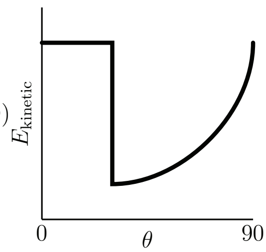
    (A) 71 s
    (B) 100 s
    (C) 141 s
    (D) 1000 s
    (E) ∞ (the ball never comes to a rest)

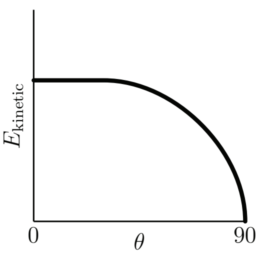
22. A solid ball is released from rest down inclines of various inclination angles θ but through a fixed vertical height h. The coefficient of static and kinetic friction are both equal to μ. Which of the following graphs best represents the total kinetic energy of the ball at the bottom of the incline as a function of the angle of the incline?

(A) 

(B) 

(C) 

(D) 

(E) 

23. A 2.0 kg object falls from rest a distance of 5.0 meters onto a 6.0 kg object that is supported by a vertical massless spring with spring constant $k = 72$ N/m. The two objects stick together after the collision, which results in the mass/spring system oscillating. What is the maximum magnitude of the displacement of the 6.0 kg object from its original location before it is struck by the falling object?

(A) 0.27 m
(B) 1.1 m
(C) 2.5 m
(D) 2.8 m
(E) 3.1 m

24. The speed of a transverse wave on a long cylindrical steel string is given by

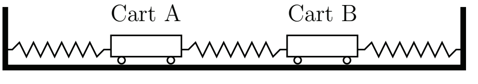
$$v = \sqrt{\frac{T}{M/L}},$$

where T is the tension in the string, M is the mass, and L is the length of the string. Ignore any string stiffness, and assume that it does not stretch when tightened.

Consider two steel strings of the same length, the first with radius r_1 and a second thicker string with radius $r_2 = 4r_1$. Each string is tightened to the maximum possible tension without breaking. What is the ratio f_1/f_2 of the fundamental frequencies of vibration on the two strings?

(A) 1
(B) $\sqrt{2}$
(C) 2
(D) $2\sqrt{2}$
(E) 4

25. Two identical carts A and B each with mass m are connected via a spring with spring constant k. Two additional springs, identical to the first, connect the carts to two fixed points. The carts are free to oscillate under the effect of the springs in one dimensional frictionless motion.



Under suitable initial conditions, the two carts will oscillate in phase according to

$$x_{\mathrm{A}}(t) = x_0 \sin \omega_1 t = x_{\mathrm{B}}(t)$$

where x_{A} and x_{B} are the locations of carts A and B relative to their respective equilibrium positions.

Under other suitable initial conditions, the two carts will oscillate exactly out of phase according to

$$x_{\mathrm{A}}(t) = x_0 \sin \omega_2 t = -x_{\mathrm{B}}(t)$$

Determine the ratio ω_2/ω_1

(A) $\sqrt{3}$
(B) 2
(C) $2\sqrt{2}$
(D) 3
(E) 5

Answers, Problem Difficulty, and Topics

ToC

2015.1	D	⋆	linear motion
2015.2	A	⋆	linear motion
2015.3	B	⋆ ⋆	power, air friction
2015.4	A	⋆ ⋆	friction (static and kinetic)
2015.5	E	⋆	projectile motion
2015.6	A	⋆	collisions, center of mass
2015.7	A	⋆	collisions
2015.8	B	⋆ ⋆	collisions
2015.9	A	⋆	collisions, conservation of linear momentum
2015.10	E	⋆ ⋆	collisions, kinetic energy
2015.11	C	⋆	Archimedes' Principle
2015.12	E	⋆ ⋆	simple pendulum
2015.13	D	⋆	simple pendulum
2015.14	C	⋆	circular motion, centripetal force
2015.15	D	⋆ ⋆	conservation of angular momentum, energy
2015.16	E	⋆	potential energy, force
2015.17	E	⋆ ⋆	dimensional analysis
2015.18	D	⋆	data analysis
2015.19	A	⋆ ⋆	oscillations, fluids
2015.20	B	⋆	forces, fluids
2015.21	B	⋆ ⋆ ⋆	linear motion
2015.22	D	⋆ ⋆ ⋆	rolling motion, friction, energy
2015.23	B	⋆ ⋆ ⋆	springs, preloading, energy
2015.24	A	⋆ ⋆ ⋆	sound, waves
2015.25	A	⋆ ⋆	springs, oscillations

Solutions

ToC

2015.1. Let us denote the width of the river by $s = 600$ m, the velocity of the boat with respect to water by $\vec{u}$ (with $u = 5$ m/s), and the velocity of the river with respect to land by $\vec{v}$ (with $v = 4$ m/s). The velocity $\vec{w}$ of the boat with respect to land is $\vec{w} = \vec{u} + \vec{v}$ and is given to be directly East, while the river flow is given to be directly South, as illustrated in Figure 1. Since vectors $\vec{w}$ and $\vec{v}$ are perpendicular,

$$w = \sqrt{u^2 - v^2} = 3 \text{ m/s}.$$

Finally, the crossing time is

$$t = \frac{d}{w} = \frac{d}{\sqrt{u^2 - v^2}} = \frac{600 \text{ m}}{3 \text{ m/s}} = 200 \text{ s} \quad \Rightarrow \quad \text{the answer is D.}$$

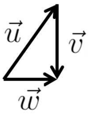


Figure 1: Illustration for Problem 2015.1.

2015.2. Let us denote the given distances and speeds as $v_1 = 80$ km/h, $s_1 = 25$ km, $v_2 = 50$ km/h, and $s_2 = 75$ km. Average speed v is total distance divided by total time, so

$$v = \frac{s_1 + s_2}{t_1 + t_2} = \frac{s_1 + s_2}{\frac{s_1}{v_1} + \frac{s_2}{v_2}} = 55.17 \text{ km/h} \quad \Rightarrow \quad \text{the answer is A.}$$

2015.3. The sustained (terminal) speed at a given power level is a stable equilibrium, a result of balanced engine and friction forces. If the speed was a bit smaller than the sustained speed, the friction would be smaller than the engine force and the plane would accelerate. If the speed was a bit higher than the sustained speed, the friction would be higher than the engine force and the plane would slow down. Therefore, at the sustained speed, the friction force is equal to the engine force, so

$$P = Fv = F_{fric}v = kv^3 \quad \Rightarrow \quad \frac{v_1}{v_0} = \sqrt[3]{\frac{P_1}{P_0}} = \sqrt[3]{2} = 1.26 \quad \Rightarrow \quad \text{the answer is B.}$$

2015.4. If $\mu_k = c\mu_s$ (where $0 \le c \le 1$) and force $P > \mu_s mg$, Newton's Second Law is

$$ma = P - \mu_k mg = P - c\mu_s mg$$

and

$$a = \frac{P}{m} - c\mu_s g.$$

We see that for $P > \mu_s mg$, acceleration a is a linear function of P with slope $1/m$ and intercept $-c\mu_s g$. From this we conclude

- B describes the solution for $c = 0$
- C describes the solution for $c = 1$
- A describes the solution for $c = 0.9$
- D and E do not make much sense in this problem

Therefore, the answer is A.

2015.5. Let $m = 470$ g, $\theta = 60^o$, $v_0 = 100$ m/s, and $d = 150$ m. The time of flight can be obtained from the horizontal distance $d\cos\theta$ and horizontal speed $v_0\cos\theta$,

$$t = \frac{d\cos\theta}{v_0\cos\theta} = \frac{d}{v_0} = 1.5 \text{ s}.$$

If there was no gravity, the projectile would hit the target. With gravity, it undergoes an additional vertical displacement (which is really the miss distance we are after) in the amount of

$$h = \frac{gt^2}{2} = \frac{gd^2}{2v_0^2} = 11.25 \text{ m}.$$

The closest available answer to this value is 11 m, i.e., E.

Note 1: Why can't we use the time-of-flight formula

$$T = \frac{2v_0\sin\theta}{g} \quad ?$$

It is because that formula is derived assuming the projectile ends its flight at the same level at which it was launched. There is no indication that is the case, indeed, $T = 17.32$ s tells us it is really not the case.

Note 2: Mass was an extraneous piece of information - it was not needed in this problem!

Note 3: This problem is a variant of the famous problem involving a monkey and a hunter[1].

[1] `https://en.wikipedia.org/wiki/The_Monkey_and_the_Hunter`

2015.6. We could solve this problem by calculating what happens with the center of mass after each collision, but there is a much simpler solution based on the fact that the velocity of the center of mass of the system does not change regardless of what kinds of collisions the carts undergo. This is a consequence of the law of conservation of linear momentum, which is valid in all types of collisions, from perfectly elastic to perfectly inelastic.

Thus, the velocity of the center of mass of this system after all collisions will be the same as it was before the first collision (note that $v_2 = -2.5$ m/s):

$$v = \frac{m_1 v_1 + m_2 v_2 + m_3 v_3}{m_1 + m_2 + m_3} = 0.11 \text{ m/s} \quad \Rightarrow \quad \text{the answer is A.}$$

2015.7. Completely (or perfectly) inelastic means that

- Objects merge
- Only conservation of linear momentum applies.

With this, for the linear momentum before the collisions, p_0, and the linear momentum after the collisions, p_1, we can write

$$p_0 = mv_0 \qquad \text{and} \qquad p_1 = (1 + 3 + 9)mv_1.$$

The law of conservation of linear momentum tells us that $p_1 = p_0$, so $v_1 = v_0/13$, which is answer A.

2015.8. In elastic collisions both the linear momentum and the kinetic energy are conserved. If the two masses colliding are m_1 and m_2 and their respective velocities before and after the collision are v_1, v_2, w_1, and w_2, we can write

$$m_1 v_1 + m_2 v_2 = m_1 w_1 + m_2 w_2 \quad \text{and} \quad \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 w_1^2}{2} + \frac{m_2 w_2^2}{2}$$

These equations imply $v_1 + w_1 = v_2 + w_2$, and finally

$$w_1 = \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} \qquad \text{and} \qquad w_2 = \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2}.$$

The collision of A and B gives us (using the formula for w_2 with $v_1 = v_0$ and $v_2 = 0$)

$$v_B = \frac{v_0}{2}.$$

The collision of B and C gives us (using the formula for w_2 with $v_1 = v_B = \frac{v_0}{2}$ and $v_2 = 0$)

$$v_C = \frac{v_0}{4} \quad \Rightarrow \quad \text{the answer is B.}$$

2015.9. Before the collision, the linear momentum of the system was

$$p_0 = 0.65 \text{ kg} \cdot 5 \text{ m/s}.$$

After the collision it remains the same (conservation of linear momentum), so we can write

$$\vec{p}_0 = \vec{p}_1 + \vec{p}_2.$$

Since $\vec{p}_1$ and $\vec{p}_2$ are perpendicular, the magnitude of their sum can be calculated through the Pythagorean theorem:

$$p_0^2 = p_1^2 + p_2^2,$$

so

$$(0.65 \cdot 5)^2 = (0.65v_1)^2 + (0.75 \cdot 4)^2 \quad \Rightarrow \quad v_1 = 1.92 \text{ m/s},$$

and the answer is A.

2015.10. In this and previous problem we do not know whether the collision is elastic or not. Speaking in general, the change of kinetic energy $\Delta K = K_f - K_i$ cannot be positive, therefore answers in A, B, and C are incorrect. If the collision is elastic, the answer will be D, and if it is inelastic, the answer will be E.

So, how do we determine which case we have here? Let us plug in the numbers and check if the kinetic energies before and after the collision are equal or not:

$$K_i = \frac{0.65 \cdot 5^2}{2} = 8.125 \text{ J} \quad \text{and} \quad K_f = \frac{0.65 \cdot 1.92^2}{2} + \frac{0.75 \cdot 4^2}{2} = 7.20 \text{ J}.$$

Since $K_i \neq K_f$, we have an inelastic collision, and the answer is E.

2015.11. Let the densities of the sphere, water, and oil be ρ, ρ_w, and ρ_o, respectively. The fraction of the sphere submerged in water is

$$x = \frac{\rho}{\rho_w} = \frac{2}{3},$$

while the fraction of the sphere submerged in oil is

$$y = \frac{\rho}{\rho_o} = \frac{\rho}{\frac{3}{4}\rho_w} = \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}.$$

Therefore, the answer is C.

2015.12. In general, for a pendulum with string length ℓ and angle amplitude θ_0, the vector of total acceleration of the pendulum bob, just like for any object on a circular trajectory, has two components:

- Centripetal (radial) component is there as in any circular motion (here it is 0 only when $|\theta| = \theta_0$):

$$a_c = \frac{v^2}{\ell} = \frac{2g\Delta h}{\ell} = \frac{2g\ell(\cos\theta - \cos\theta_0)}{\ell} = 2g(\cos\theta - \cos\theta_0)$$

- Tangential, due to the tangential component of gravity (here it is 0 only when $\theta = 0$): $a_t = mg\sin\theta$.

The magnitude of total acceleration is $a = \sqrt{a_c^2 + a_t^2}$.

Figure 2 shows the vectors of total acceleration and velocity for a half of a full pendulum period:

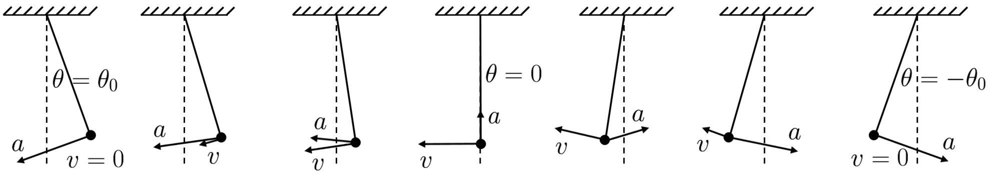


Figure 2: Illustration for Problem 2015.12. A nice GIF animation can be seen here: https://en.wikipedia.org/wiki/File:Oscillating_pendulum.gif.

What happens when the bob is at the lowest point? The radial component is (as always) directed to the center of rotation, so it is vertical when the bob is at the bottom. At the same time, the tangential acceleration is 0 because all forces on the bob are in vertical direction (tension of the rope pulls it up while gravity pulls it down). That is why the correct answer is E.

Note: It is instructive to see what is wrong with other answers.

- A is not correct because the magnitude of total acceleration of the bob changes as it moves (see Figure 2).
- B is not correct because at the lowest point the total acceleration has only the radial component and is pointed up.
- C is not correct because a_c and a_t are zero for different values of θ.
- D is not correct because (except at the bottom) there is always a tangential component of gravity $g\sin\theta$.

2015.13. Decompose given vectors into radial and tangential components. The radial component of acceleration is always directed towards the center of rotation, therefore E is out. Tangential component is due to tangential component of gravity and points towards the equilibrium, regardless of direction of bob's motion, therefore, A, B, and C are out. The answer is D.

2015.14. Let $D = 3$ m, $m = 5$ kg, and $\omega = 4$ rad/s. Let $R = D/2$ be the radius of rotation. Because the rotation is in a horizontal plane, the weight of the objects is not a factor. The tension force T keeps the objects on a circular trajectory, so it is the centripetal force:

$$T = m\omega^2 R = \frac{m\omega^2 D}{2} = 120 \text{ N}.$$

Another way to look at this problem is from the reference system of one of the objects, when we can say that the tension force T balances the centrifugal force $m\omega^2 R$, etc. Either way, the answer is C.

2015.15. Again, let $D = 3$ m, $m = 5$ kg, and $\omega = 4$ rad/s, and let $R = D/2$ be the radius of rotation. Let also $r = 0.5$ m be the final radius of rotation. The angular momentum L is conserved because there are no external forces and torques acting on the system. That is a very useful observation because as the objects are pulled in, their angular speed changes, so we will express the kinetic energy of the system using L as an invariant. Analogous to how in linear motion

$$E_k = \frac{mv^2}{2} = \frac{p^2}{2m} \qquad (p = mv),$$

in rotational motion

$$E_k = \frac{I\omega^2}{2} = \frac{L^2}{2I} \qquad (L = I\omega),$$

where initially $I_1 = 2mR^2$ and eventually $I_2 = 2mr^2$. Then the work equation becomes

$$\begin{aligned} W &= E_{k2} - E_{k1} = \frac{L^2}{2I_2} - \frac{L^2}{2I_1} = \frac{L^2}{2}\left(\frac{1}{I_2} - \frac{1}{I_1}\right) \\ &= \frac{(2mR^2\omega)^2}{2}\left(\frac{1}{2mr^2} - \frac{1}{2mR^2}\right) \\ &= mR^2\omega^2\left(\frac{R^2}{r^2} - 1\right) \\ &= 1440 \text{ J}. \end{aligned}$$

Thus, the answer is D.

2015.16. The first four answers are true:

- A is true because the peaks of potential energy represent equilibrium points[2]. Note that a maximum of potential energy is an unstable equilibrium, while a minimum of potential energy is a stable equilibrium.
- B is obviously true.
- C is true because the force is the negative derivative (i.e., the negative slope) of the potential energy, so the magnitude of the force is equal to the absolute value of the slope of the curve. From the graph, we can estimate the slope to be
$$\frac{10\ \mathrm{J}}{1\ \mathrm{cm}} = 1000\ \mathrm{J/m} = 1000\ \mathrm{N}.$$
- D is true because total energy is $E = E_k + E_p$.

The only untrue statement is E, because the acceleration at $x = 2$ cm can be estimated to be

$$a = F/m = \frac{10\ \mathrm{J}/0.005\ \mathrm{m}}{0.5\ \mathrm{kg}} = 4000\ \mathrm{m/s^2} = 4 \cdot 10^5\ \mathrm{cm/s^2}.$$

Therefore, the desired answer is E.

2015.17. In almost every $F{=}ma$ contest there seems to be a very hard problem that is easy to solve if you notice that you can use dimensional analysis. This was one of them.

Answers D and E are the only solutions with good units. For example, in E:

$$\alpha\sigma\rho^{-1} = \ \mathrm{Pa}\left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)^{-1} = \frac{\mathrm{N}}{\mathrm{m}^2}\ \frac{\mathrm{m}^3}{\mathrm{kg}} = \frac{\mathrm{J}}{\mathrm{kg}}.$$

If you are stuck here, it is already OK to guess (especially if there is no penalty for incorrect answers as was the case this year), but we can reason that h should not be a factor in the result, as the maximum possible rotation speed before the flywheel disintegrates should not depend on its thickness, certainly not linearly, so the answer is E.

2015.18. The log-linear graph is a straight line, so we can write:

$$\ln y = kx + n \quad \Rightarrow \quad y = e^{kx+n},$$

which can be rewritten as $y = ae^{bx}$ (with $a = e^n$ and $b = k$) and not as other functions on offer, so the answer is D.

[2] The force on the object is the negative derivative of the potential, $F = -\frac{d}{dx}E_p(x)$, so at peaks the force is 0.

2015.19. We show two different approaches here:

Approach 1: This approach is a combination of science and problem-solving experience. The science we use here is the dimensional analysis, and it tells us that answers C, D, and E are out because the formulas in them do not produce the physical unit of frequency (which is hertz, Hz = 1/s). At this point you may want to make a guess (this year was penalty-free) but with some problem-solving experience and after seeing many formulas for periods of oscillation, you will know that the period of oscillations T usually has the factor 2π in the formula, therefore, $f = 1/T$ usually has the factor $\frac{1}{2\pi}$, and we see that the answer is A.

Approach 2: A fully scientific approach would be to set up a differential equation for this system, or at least to use one of the two equivalent tricks that do the same thing while avoiding calculus terminology:

1. If displacement x causes acceleration a, then the period of oscillations is

$$T = 2\pi\sqrt{\frac{x}{a}}$$

In our problem, if the water level on one side of the U tube is lowered by displacement x, it will go up by the same displacement x. Let the area of the cross-section of the U-tube be A. The restoring force is due to the height of $2x$ of water, and is equal to its weight:

$$F = m_{2x}g = \rho_w V_{2x} g = 2\rho_w x A g.$$

This force accelerates the entire mass of water in the U-tube:

$$m = \rho_w V = \rho_w L A.$$

Finally, the acceleration corresponding to displacement x is

$$a = F/m = \frac{2xg}{L},$$

so the period of oscillation is

$$T = 2\pi\sqrt{\frac{x}{a}} = 2\pi\sqrt{\frac{L}{2g}}$$

while

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{2g}{L}}$$

so the answer is A.

2. If the system has mass m and an effective spring constant k, then

$$T = 2\pi\sqrt{\frac{m}{k}}$$

In our case, the mass of the system is

$$m = \rho_w V = \rho_w LA.$$

The effective spring constant is the ratio of the restoring force F and the displacement x:

$$k = \frac{F}{x} = \frac{2\rho_w xAg}{x} = 2\rho_w Ag.$$

Finally,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{L}{2g}} \qquad \Rightarrow \qquad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{2g}{L}}$$

so the answer is A.

2015.20. Let z be the height of water on the side of the U-tube with oil. Then, the height of water on the other side is $L - z$ and the height difference is $h = (L + z) - (L - z) = 2z$. In equilibrium, static pressures of two columns of fluid must be equal, so we can write (also recall that $\rho_o = \rho_w/2$)

$$\rho_o gL + \rho_w gz = \rho_w g(L - z) \quad \Rightarrow \quad z = \frac{\rho_w - \rho_o}{2\rho_w}L = \frac{L}{4} \quad \Rightarrow \quad y = 2z = \frac{L}{2},$$

so the answer is B.

2015.21. Let $v_0 = 50$ m/s. The time of flight between n-th and $(n + 1)$-th bounce $(n = 1, 2, \ldots)$ is

$$T_n = \frac{2v_n}{g}, \qquad \text{where} \qquad v_n = C^n v_0.$$

This includes the original flight, right after the launch, with index $n = 0$. The total time of this entire experiment is then

$$T = \sum_{n=0}^{\infty} T_n = \sum_{n=0}^{\infty} \frac{2v_n}{g} = \frac{2v_0}{g}\sum_{n=0}^{\infty} C^n.$$

Here we use the formula for the sum of a geometric series:

$$1 + a + a^2 + \ldots = \frac{1}{1-a} \qquad \text{(valid only for } |a| < 1).$$

Here $|C| < 1$, therefore

$$T = \frac{2v_0}{g(1 - C)} = 100 \text{ s} \quad \Rightarrow \quad \text{the answer is B.}$$

2015.22. For angles θ below some critical angle[3] θ_c, the ball rolls without slipping, therefore the friction is static. Since static friction does not dissipate energy, for $0 < \theta < \theta_c$ no energy is lost, so the correct graph will have a constant section, just like graphs in C, D, and E. When $\theta = 90^o$, the body will be in free fall, so again no energy is lost to friction and the correct graph will have the same value as the initial constant section. In other words, E is out and we are left with C and D as the remaining possibilities. Since $\mu_k = \mu_s$, the slipping gradually increases as θ is increased from θ_c, so we should not expect to see a discontinuity in the resulting E_k vs. θ graph, i.e., result D is out and we are left with the final result, C.

Note 1: Here we show a full derivation of the dependence of E_k on θ. We already determined that the critical angle θ_c is such that

$$\tan\theta_c = \frac{7}{2}\mu$$

and that for $0 < \theta < \theta_c$ the total kinetic energy at the bottom of the slope is constant. It will be equal to the potential energy of the ball at the top of the slope, $E_k = E_p = mgh$.

For angles beyond θ_c the friction will be kinetic, $F_k = \mu mg\cos\theta$, and so linear and angular speeds will be

$$v = at = \frac{Ft}{m} = \frac{(F_p - F_k)t}{m} = \frac{(mg\sin\theta - \mu mg\cos\theta)t}{m} = gt(\sin\theta - \mu\cos\theta)$$

$$\omega = \alpha t = \frac{\tau t}{I} = \frac{F_k rt}{I} = \frac{\mu mgtr\cos\theta}{\frac{2}{5}mr^2} = \frac{5\mu gt\cos\theta}{2r}.$$

Let us determine the time at which the ball reaches the end of the slope:

$$s = \frac{at^2}{2} = \frac{h}{\sin\theta} \quad\Rightarrow\quad t = \sqrt{\frac{2h}{a\sin\theta}} = \sqrt{\frac{2h}{g\sin\theta(\sin\theta - \mu\cos\theta)}}.$$

[3] This critical angle can be found by considering the boundary case for static friction. From

$$ma = mg\sin\theta - F_s \qquad\text{and}\qquad I\alpha = F_s r$$

we get

$$a = g\sin\theta - \frac{F_s}{m} \qquad\text{and}\qquad \alpha = \frac{F_s r}{I}.$$

With static friction there is no slipping, so $a = \alpha r$, and we get

$$F_s = \frac{mI\sin\theta}{I + mr^2}.$$

Since $F_s \le F_s^{max} = \mu mg\cos\theta$, the critical angle for which F_s reaches its limit can be found from

$$\frac{mI\sin\theta_c}{I + mr^2} = \mu mg\cos\theta_c \quad\Rightarrow\quad \tan\theta_c = \mu\frac{I + mr^2}{I}.$$

In this case $I = \frac{2}{5}mr^2$, so $\tan\theta_c = \frac{7}{2}\mu$. For example, if $\mu = 0.2$, then $\theta_c \approx 35^o$.

We are now ready to derive the formula for the total kinetic energy. With $I = mr^2$ and v, ω, and t as derived above, we get

$$E_k = E_k^{transl} + E_k^{rot} = \frac{mv^2}{2} + \frac{I\omega^2}{2} = mgh\,\frac{(\tan\theta - \mu)^2 + \frac{5}{2}\mu^2}{\tan\theta(\tan\theta - \mu)}.$$

A quick check for $\theta = \theta_c$ and $\theta \to 90^o$ is a good idea here. Using $\tan\theta_c = \frac{7}{2}\mu$:

$$E_k(\theta_c) = mgh\,\frac{\left(\frac{7}{2} - 1\right)^2 + \frac{5}{2}}{\frac{7}{2}\cdot\frac{5}{2}} = mgh.$$

For the case $\theta \to 90^o$ we can use substitution $x = \tan\theta$:

$$\lim_{\theta\to 90^o} E_k(\theta) = \lim_{x\to\infty} mgh\,\frac{(x-\mu)^2 + \frac{5}{2}\mu^2}{x(x-\mu)} = mgh.$$

Finally, we can write

$$E_k = \begin{cases} mgh & \text{for } 0 < \theta < \theta_c \\ mgh\,\dfrac{(\tan\theta - \mu)^2 + \frac{5}{2}\mu^2}{\tan\theta(\tan\theta - \mu)} & \text{for } \theta_c \le \theta < 90^o \\ mgh & \text{for } \theta = 90^o. \end{cases}$$

For example, if $\mu = 0.2$ and $mgh = 1$ J, the plot will look like Figure 3:

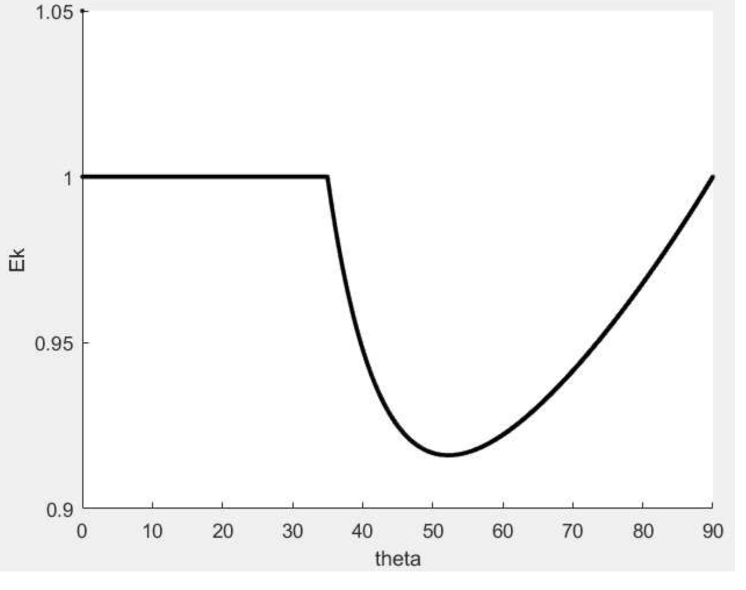


Figure 3: Illustration for Problem 2015.22.

Note 2: Another way to derive the formula for E_k for $\theta_c < \theta < 90^o$ is by using the law of conservation of energy:

$$E_k = E_p - W,$$

where $E_p = mgh$ and W is the work done by the kinetic friction. In principle, this is a simpler approach, but we have to be careful about the displacement over which the kinetic friction acts. The full length of the slope $s = \frac{h}{\sin\theta}$ is shortened due to rolling. The shortening is

$$\Delta s = r\Delta\phi = r\,\frac{\alpha t^2}{2} = \frac{5\mu gh\cos\theta}{2g\sin\theta(\sin\theta - \mu\cos\theta)}.$$

The effective length for the work of kinetic friction is then

$$d = s - \Delta s = \frac{h(\sin\theta - \frac{7}{2}\cos\theta)}{\sin\theta(\sin\theta - \mu\cos\theta)},$$

and so for $\theta_c < \theta < 90^o$

$$E_k = E_p - W = mgh\,\frac{(\tan\theta - \mu)^2 + \frac{5}{2}\mu^2}{\tan\theta(\tan\theta - \mu)}.$$

2015.23. In this problem it is important to notice that the spring was already compressed before the collision, so an additional compression x is *not* going to add additional spring energy $\frac{1}{2}kx^2$. Rather, if the original compression was by amount h, the change in spring energy after additional compression x will be

$$\Delta E_s = \frac{1}{2}k(h+x)^2 - \frac{1}{2}kh^2 = \frac{1}{2}kx^2 + khx.$$

The term khx is due to the quadratic dependence of the spring's energy on its compression.

Let $m_1 = 2$ kg, $y = 5$ m, $m_2 = 6$ kg, and $k = 72$ N/m. Let us first determine the original compression h of the spring, before the collision. Let "up" be the positive direction. Since m_2 was originally in equilibrium, its weight is balanced by the restoring force of the spring $F_s = -kh$

$$F_s - m_2 g = 0 \quad\Rightarrow\quad -kh = m_2 g \quad\Rightarrow\quad h = -\frac{m_2 g}{k} = -0.833 \text{ m}.$$

Next, we can get the speed of m_1 just before the collision with m_2 from conservation of energy:

$$mgy = \frac{mv^2}{2} \qquad\Rightarrow\qquad v = \sqrt{2gy} = 10 \text{ m/s}.$$

The collision is completely inelastic so the speed of the merged object is

$$w = \frac{m_1 v}{m_1 + m_2} = \frac{m_1\sqrt{2gy}}{m_1 + m_2} = 2.5 \text{ m/s}.$$

Finally, we use conservation of energy between the moment right after the collision and the moment when the merged body is at its lowest point, i.e., at displacement $x < 0$ with respect to the position of m_2 before the collision.

The total energy $E^{(1)}$ right after the collision consisted of

- Kinetic energy $E_k^{(1)} = \frac{(m_1+m_2)w^2}{2}$
- Spring potential energy $E_s^{(1)} = \frac{1}{2}kh^2$
- Gravitational potential energy $E_p^{(1)} = 0$ (i.e., we choose this to be the zero-level)

The total energy $E^{(2)}$ at the moment the merged object is at its lowest point (with displacement $x < 0$ from the original position and speed 0) is

- Kinetic energy $E_k^{(2)} = 0$
- Spring potential energy $E_s^{(2)} = \frac{1}{2}k(h + x)^2$
- Gravitational potential energy $E_p^{(2)} = (m_1 + m_2)gx$ (we are now below zero-level, so $x < 0$ and $E_p^{(2)} < 0$)

Finally, from $E^{(1)} = E^{(2)}$ we find

$$\frac{(m_1 + m_2)w^2}{2} + \frac{1}{2}kh^2 = \frac{1}{2}k(h + x)^2 + (m_1 + m_2)gx$$

$$\frac{(m_1 + m_2)w^2}{2} + \cancel{\frac{1}{2}kh^2} = \cancel{\frac{1}{2}kh^2} + khx + \frac{1}{2}kx^2 + (m_1 + m_2)gx.$$

We can simplify this further by noticing that $m_2 g = -kh$:

$$\frac{(m_1 + m_2)w^2}{2} = \frac{1}{2}kx^2 + m_1 gx. \tag{1}$$

Equation (1) is quadratic in displacement x with only one negative solution:

$$36x^2 + 20x - 25 = 0 \qquad \Rightarrow \qquad x = -1.156 \text{ m}.$$

The problem is asking for the magnitude (absolute value) of this displacement, so the best answer is B.

Note 1: Had we not taken into account the quadratic nature of the spring's potential energy, we would naively write an equation very similar to Equation (1), except that in the right-most term we would have $m_1 + m_2$ instead of just m_1. When done properly, it is as if m_2 is weightless – its weight is balanced by the spring.

Note 2: A very interesting question about this problem was asked by a student in our class: Couldn't we simply consider the balance of the spring force and gravity, $(m_1+m_2)g = kx$, which yields $x = (m_1+m_2)g/k = 1.11$ m, the correct answer? Unfortunately, the answer is no, because balancing of forces works to find the equilibrium positions, and here we are dealing with the extreme position of oscillations, when the forces are not balanced. The only approach that works here is the conservation of energy. The fact that the wrong approach produced a correct result was a pure coincidence. If the given values were different, for example, if $m_2 = 1$ kg, the two results would be quite different. This can be seen from the two formulas resulting from the two approaches:

$$|x| = \frac{m_1 g}{k}\left(1+\sqrt{1+\frac{2ky}{(m_1+m_2)g}}\right) \qquad \text{and} \qquad |x| = \frac{(m_1+m_2)g}{k}.$$

2015.24. We show two different approaches here:

Approach 1: This approach is based on experience and intuition. If you ever played (or played with) a guitar or a violin and broke a few strings in the process by trying to tighten them too much, you will probably know that the sound produced by extremely tight strings depends only on their length, not on their thickness. Therefore, the right answer is A.

Approach 2: This approach is based on the fact that a string breaks when tension $T > \sigma A$, where σ is the tensile strength and A is the area of the string's cross section. Then, for $T = \sigma A$, the speed of sound in the string will be

$$v = \sqrt{\frac{TL}{M}} = \sqrt{\frac{\sigma A L}{M}} = \sqrt{\frac{\sigma}{\rho}},$$

where ρ is the density of the string. This expression shows that when the tension is at its maximum, the speed of sound in the string depends only on the properties of the material, not on its geometry (length or thickness). Since $v = f\lambda$ and for the fundamental oscillation of a string fixed at both ends we have $\lambda = 2L$, we find that the fundamental frequency of a fully tightened string is

$$f = \frac{2v}{L} = \frac{2}{L}\sqrt{\frac{\sigma}{\rho}},$$

and for two strings that differ only in thickness, their fundamental frequencies are equal. Therefore, the right answer is A.

2015.25. When the carts are in-phase, the distance between them is constant, so the force of A on B is constant. In that case, only one spring exerts a varying force on each cart, and then the angular frequency is given by

$$\omega_1 = \sqrt{\frac{k}{m}}.$$

When the carts move with opposite phases, the distance between the carts is not constant, but the center of the connecting spring does not move. Therefore, either cart can be though of as moving under the influence of two springs in parallel, one with length L, the other with length $L/2$. That is equivalent to having a spring with constant k in parallel with a spring of constant $2k$. The equivalent spring constant is then $k' = k + 2k = 3k$, so

$$\omega_2 = \sqrt{\frac{3k}{m}}$$

and

$$\frac{\omega_2}{\omega_1} = \sqrt{3} \qquad \Rightarrow \qquad \text{the answer is A.}$$

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Year 2016

$F = ma$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2016

2016 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

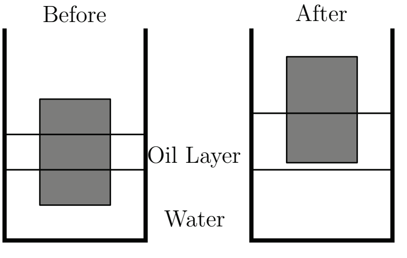
- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2016.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

1. A car drives anticlockwise (counterclockwise) around a flat, circular curve at constant speed, so that the left, front wheel traces out a circular path of radius $R = 9.60$ m. If the width of the car is 1.74 m, what is the ratio of the angular velocity about its axle of the left, front wheel to that of the right, front wheel, of the car as it moves through the curve? Assume the wheels roll without slipping.

   (A) 0.331
   (B) 0.630
   (C) 0.708
   (D) 0.815
   (E) 0.847

2. A 3.0 cm thick layer of oil with density $\rho_o = 800$ kg/m^3 is floating above water that has density $\rho_w = 1000$ kg/m^3. A solid cylinder is floating so that 1/3 is in the water, 1/3 is in the oil, and 1/3 is in the air. Additional oil is added until the cylinder is floating only in oil. What fraction of the cylinder is in the oil?

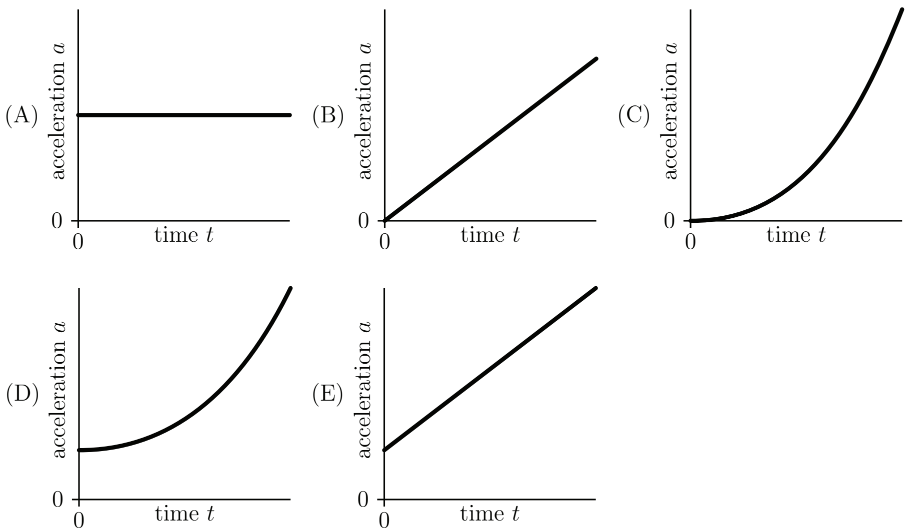


   (A) 3/5
   (B) 3/4
   (C) 2/3
   (D) 8/9
   (E) 4/5

3. An introductory physics student, elated by a first semester grade, celebrates by dropping a textbook from a balcony into a deep layer of soft snow which is 3.00 m below. Upon hitting the snow the book sinks a further 1.00 m into it before coming to a stop. The mass of the book is 5.0 kg. Assuming a constant retarding force, what is the force from the snow on the book?

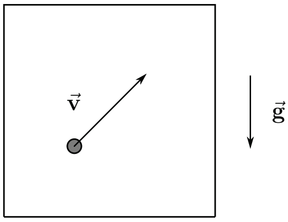
   (A) 85 N
   (B) 100 N
   (C) 120 N
   (D) 150 N
   (E) 200 N

4. A small bead slides from rest along a wire that is shaped like a vertical uniform helix (spring). Which graph below shows the magnitude of the acceleration a as a function of time?



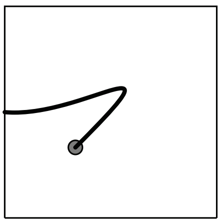
The following information applies to questions 5 and 6.

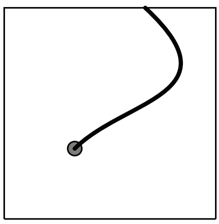
Consider a particle in a box where the force of gravity is down as shown in the figure.

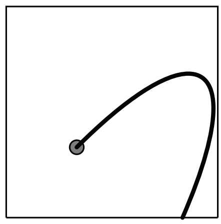


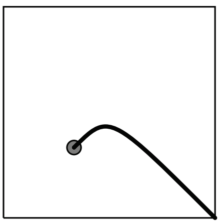
The particle has an initial velocity as shown, and the box has a constant acceleration to the right.

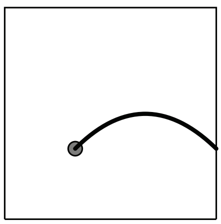
5. In the frame of the box, which of the following is a possible path followed by the particle?

(A) 

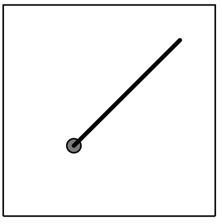
(B) 

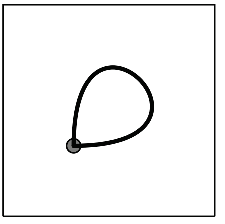
(C) 

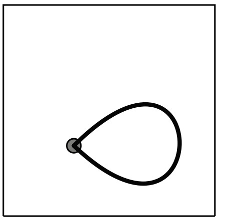
(D) 

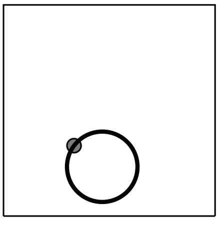
(E) 

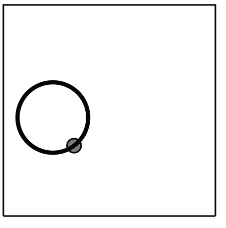
6. If the magnitude of the acceleration of the box is chosen correctly, the launched particle will follow a path that returns to the point that it was launched. In the frame of the box, which path is followed by the particle?

(A) 

(B) 

(C) 

(D) 

(E) 

7. A mass on a frictionless table is attached to the midpoint of an originally unstretched spring fixed at the ends. If the mass is displaced a distance A parallel to the table surface but perpendicular to the spring, it exhibits oscillations. The period T of the oscillations

   (A) does not depend on A.
   (B) increases as A increases, approaching a fixed value.
   (C) decreases as A increases, approaching a fixed value.
   (D) is approximately constant for small values of A, then increases without bound.
   (E) is approximately constant for small values of A, then decreases without bound.

8. Kepler's Laws state that

    I. the orbits of planets are elliptical with one focus at the sun,

    II. a line connecting the sun and a planet sweeps out equal areas in equal times, and

    III. the square the period of a planet's orbit is proportional to the cube of its semimajor axis.

    Which of these laws would remain true if the force of gravity were proportional to $1/r^3$ rather than $1/r^2$?

    (A) Only I.

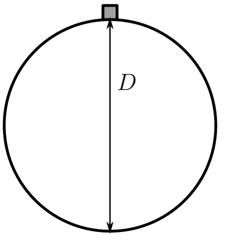
    (B) Only II.

    (C) Only III.

    (D) Both II and III.

    (E) None of the above.

9. A small bead is placed on the top of a frictionless glass sphere of diameter D as shown. The bead is given a slight push and starts sliding down along the sphere. Find the speed v of the bead at the point at which the which the bead leaves the sphere.



    (A) $v = \sqrt{gD}$

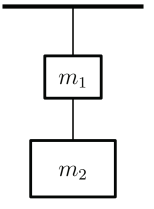
    (B) $v = \sqrt{4gD/5}$

    (C) $v = \sqrt{2gD/3}$

    (D) $v = \sqrt{gD/2}$

    (E) $v = \sqrt{gD/3}$

10. Two blocks are suspended by two massless elastic strings to the ceiling as shown in the figure. The masses of the upper and lower block are $m_1 = 2$ kg and $m_2 = 4$ kg respectively. If the upper string is suddenly cut just above the top block what are the accelerations of the two blocks at the moment when the top block begins to fall?



(A) upper: 10 m/s^2; lower: 0
(B) upper: 10 m/s^2; lower: 10 m/s^2
(C) upper: 20 m/s^2; lower: 10 m/s^2
(D) upper: 30 m/s^2; lower: 0
(E) upper: 30 m/s^2; lower: 10 m/s^2

11. The power output from a certain experimental car design to be shaped like a cube is proportional to the mass m of the car. The force of air friction on the car is proportional to Av^2, where v is the speed of the car and A the cross sectional area. On a level surface the car has a maximum speed v_{max}. Assuming that all versions of this design have the same density, then which of the following is true?

(A) $v_{\text{max}} \propto m^{1/9}$
(B) $v_{\text{max}} \propto m^{1/7}$
(C) $v_{\text{max}} \propto m^{1/3}$
(D) $v_{\text{max}} \propto m^{2/3}$
(E) $v_{\text{max}} \propto m^{3/4}$

12. A block floats partially submerged in a container of liquid. When the entire container is accelerated upward, which of the following happens? Assume that both the liquid and the block are incompressible.

(A) The block descends down lower into the liquid.
(B) The block ascends up higher in the liquid.
(C) The block does not ascend nor descend in the liquid.
(D) The answer depends on the direction of motion of the container.
(E) The answer depends on the rate of change of the acceleration

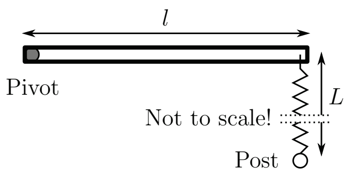
13. An object of mass m_1 initially moving at speed v_0 collides with an originally stationary object of mass $m_2 = \alpha m_1$, where $\alpha < 1$. The collision could be completely elastic, completely inelastic, or partially inelastic. After the collision the two objects move at speeds v_1 and v_2. Assume that the collision is one dimensional, and that object one cannot pass through object two.

    After the collision, the speed ratio $r_2 = v_2/v_0$ of object 2 is bounded by

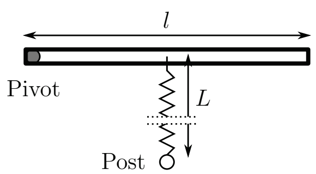
    (A) $(1-\alpha)/(1+\alpha) \le r_2 \le 1$
    (B) $(1-\alpha)/(1+\alpha) \le r_2 \le 1/(1+\alpha)$
    (C) $\alpha/(1+\alpha) \le r_2 \le 1$
    (D) $0 \le r_2 \le 2\alpha/(1+\alpha)$
    (E) $1/(1+\alpha) \le r_2 \le 2/(1+\alpha)$

The following information applies to questions 14 and 15.

A uniform rod of length l lies on a frictionless horizontal surface. One end of the rod is attached to a pivot. An un-stretched spring of length $L \gg l$ lies on the surface perpendicular to the rod; one end of the spring is attached to the movable end of the rod, and the other end is attached to a fixed post. When the rod is rotated slightly about the pivot, it oscillates at frequency f.

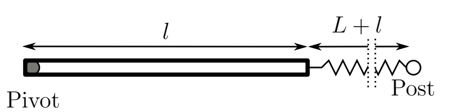


14. The spring attachment is moved to the midpoint of the rod, and the post is moved so the spring remains unstretched and perpendicular to the rod. The system is again set into small oscillations. What is the new frequency of oscillation?



(A) $f/2$
(B) $f/\sqrt{2}$
(C) f
(D) $\sqrt{2}f$
(E) $2f$

15. The spring attachment is moved back to the end of the rod; the post is moved so that it is in line with the rod and the pivot and the spring is unstretched. The post is then moved away from the pivot by an additional amount l. What is the new frequency of oscillation?

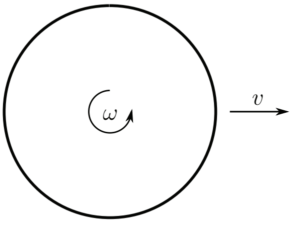


(A) $f/3$
(B) $f/\sqrt{3}$
(C) f
(D) $\sqrt{3}f$
(E) $3f$

16. A ball rolls from the back of a large truck traveling 10.0 m/s to the right. The ball is traveling horizontally at 8.0 m/s to the left relative to an observer in the truck. The ball lands on the roadway 1.25 m below its starting level. How far behind the truck does it land?

    (A) 0.50 m
    (B) 1.0 m
    (C) 4.0 m
    (D) 5.0 m
    (E) 9.0 m

17. As shown in the figure, a ping-pong ball with mass m with initial horizontal velocity v and angular velocity ω comes into contact with the ground. Friction is not negligible, so both the velocity and angular velocity of the ping-pong ball changes. What is the critical velocity v_c such that the ping-pong will stop and remain stopped? Treat the ping-pong ball as a hollow sphere.

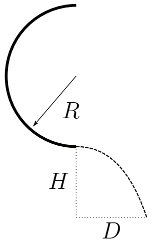


    (A) $v = \frac{2}{3}R\omega$
    (B) $v = \frac{2}{5}R\omega$
    (C) $v = R\omega$
    (D) $v = \frac{3}{5}R\omega$
    (E) $v = \frac{5}{3}R\omega$

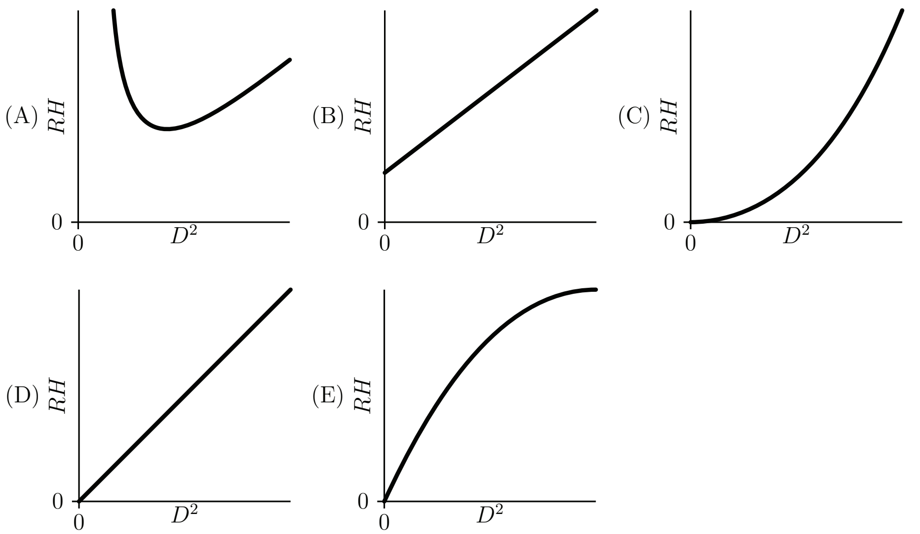
18. A spinning object begins from rest and accelerates to an angular velocity of $\omega = \pi/15$ rad/s with an angular acceleration of $\alpha = \pi/75$ rad/s^2. It remains spinning at that constant angular velocity and then stops with an angular acceleration of the same magnitude as it previously accelerated. The object made a total of 3 complete rotations during the entire motion. How much time did the motion take?

    (A) 75 s
    (B) 80 s
    (C) 85 s
    (D) 90 s
    (E) 95 s

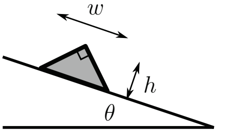
19. A semicircular wire of radius R is oriented vertically. A small bead is released from rest at the top of the wire, it slides without friction under the influence of gravity to the bottom, where is then leaves the wire horizontally and falls a distance H to the ground. The bead lands a horizontal distance D away from where it was launched.



Which of the following is a correct graph of RH against D^2?



20. A uniform solid right prism whose cross section is an isosceles right triangle with height h and width $w = 2h$ is placed on an incline that has a variable angle with the horizontal θ. What is the minimum coefficient of static friction so that the prism topples before it begins sliding as θ is slowly increased from zero?



(A) 0.71
(B) 1.41
(C) 1.50
(D) 1.73
(E) 3.00

The following information applies to questions 21 and 22.

A small ball of mass $3m$ is at rest on the ground. A second small ball of mass m is positioned above the ground by a vertical massless rod of length L that is also attached to the ball on the ground. The original orientation of the rod is directly vertical, and the top ball is given a small horizontal nudge. There is no friction; assume that everything happens in a single plane.

21. Determine the horizontal displacement x of the second ball just before it hits the ground.

(A) $x = \frac{3}{4}L$
(B) $x = \frac{3}{5}L$
(C) $x = \frac{1}{4}L$
(D) $x = \frac{1}{3}L$
(E) $x = \frac{2}{5}L$

22. Determine the speed v of the second (originally top) ball just before it hits the ground.

(A) $v = \sqrt{2gL}$
(B) $v = \sqrt{gL}$
(C) $v = \sqrt{2gL/3}$
(D) $v = \sqrt{3gL/2}$
(E) $v = \sqrt{gL/4}$

23. A uniform thin circular rubber band of mass M and spring constant k has an original radius R. Now it is tossed into the air. Assume it remains circular when stabilized in air and rotates at angular speed ω about its center uniformly. Which of the following gives the new radius of the rubber band?

    (A) $(2\pi kR)/(2\pi k - M\omega^2)$
    (B) $(4\pi kR)/(4\pi k - M\omega^2)$
    (C) $(8\pi^2 kR)/(8\pi^2 k - M\omega^2)$
    (D) $(4\pi^2 kR)/(4\pi^2 k - M\omega^2)$
    (E) $(4\pi kR)/(2\pi k - M\omega^2)$

24. The moment of inertia of a uniform equilateral triangle with mass m and side length a about an axis through one of its sides and parallel to that side is $(1/8)ma^2$. What is the moment of inertia of a uniform regular hexagon of mass m and side length a about an axis through two opposite vertices?

    (A) $(1/6)ma^2$
    (B) $(5/24)ma^2$
    (C) $(17/72)ma^2$
    (D) $(19/72)ma^2$
    (E) $(9/32)ma^2$

25. Three students make measurements of the length of a 1.50 meter rod. Each student reports an uncertainty estimate representing an independent random error applicable to the measurement.

    Alice: A single measurement using a 2.0 meter long tape measure, to within ± 2 mm.

    Bob: Two measurements using a wooden meter stick, each to within ± 2 mm, which he adds together.

    Christina: Two measurements using a machinist's meter ruler, each to within ± 1 mm, which she adds together.

    The students' teacher prefers measurements that are likely to have less error. Which is the correct order of preference?

    (A) Christina's is preferable to Alice's, which is preferable to Bob's
    (B) Alice's is preferable to Christina's, which is preferable to Bob's
    (C) Alice's and Christina's are equally preferable; both are preferable to Bob's
    (D) Christina's is preferable to both Alice's and to Bob's, which are equally preferable
    (E) Alice's is preferable to Bob's and Christina's, which are equally preferable

Answers, Problem Difficulty, and Topics

ToC

2016.1	E	⋆	circular motion
2016.2	B	⋆ ⋆	Archimedes' Principle
2016.3	E	⋆	linear motion
2016.4	D	⋆ ⋆	circular motion, composite 3D motion
2016.5	C	⋆	projectile motion
2016.6	A	⋆	projectile motion
2016.7	C	⋆ ⋆ ⋆	springs, oscillations
2016.8	B	⋆ ⋆	Kepler's Laws
2016.9	E	⋆ ⋆	centripetal force, conservation of energy
2016.10	D	⋆	forces
2016.11	A	⋆ ⋆	dimensional analysis
2016.12	C	⋆	Archimedes' Principle
2016.13	E	⋆ ⋆ ⋆	collisions
2016.14	A	⋆ ⋆	springs, oscillations
2016.15	C	⋆ ⋆ ⋆	springs, oscillations
2016.16	C	⋆	projectile motion
2016.17	A	⋆ ⋆	rolling motion, torque, angular momentum
2016.18	E	⋆	circular motion
2016.19	D	⋆ ⋆	projectile motion, conservation of energy
2016.20	E	⋆ ⋆	static friction, center of mass
2016.21	A	⋆	center of mass, conservation of linear momentum
2016.22	A	⋆	conservation of energy
2016.23	D	⋆ ⋆ ⋆	springs, Hooke's Law, centripetal force
2016.24	B	⋆ ⋆ ⋆	moment of inertia, parallel axis theorem
2016.25	A	⋆ ⋆	measurement error analysis

Solutions

ToC

2016.1. Let the angular speeds of the front wheels on the left and right of the car be ω_L and ω_R, respectively, and let z be the width of the car. Students with more experience can solve this very quickly as $\dfrac{\omega_L}{\omega_R} = \dfrac{R}{R+z} = 0.847$, because in a single revolution, the inner wheel moves the distance $2\pi R$, while the outer wheel moves the distance $2\pi(R+z)$. This immediately gives the ratio of speeds.

Those who want to take a more step-by-step approach can write it as follows

$$\frac{\omega_L}{\omega_R} = \frac{\frac{\theta_L}{t}}{\frac{\theta_R}{t}} = \frac{\theta_L}{\theta_R} = \frac{\frac{s_L}{r}}{\frac{s_R}{r}} = \frac{s_L}{s_R} = \frac{\omega R t}{\omega(R+z)t} = \frac{R}{R+z} = 0.847,$$

where θ_L and θ_R are the angles the left and the right wheels turn around their axles during the observation time t, while s_L and s_R are the distances around the circular track covered by the left and right wheels during the same time. ω is the angular speed of the car around the circular track.

Either way, the answer is E.

Note: The no-slipping condition is important here because it tells us that s_L and s_R are also the distances the wheels rotated ($s_L = r\theta_L$ and $s_R = r\theta_R$).

Generalization: How would you solve this problem if the right wheel had hit an oily patch and had $\varepsilon = 10\%$ of slipping? Hint: $\frac{\omega_L}{\omega_R} = \frac{R}{(1+\varepsilon)(R+z)}$.

2016.2. We are given $h = 3$ cm, $\rho_o = 800$ kg/m^3, and $\rho_w = 1000$ kg/m^3. The cylinder is floating, therefore its weight is matched by the force of buoyancy, which is equal to the weight of the displaced fluids (Archimedes' Principle). If the volume of the cylinder is V, then the mass of the displaced oil is $m_o = \frac{1}{3}\rho_o V$, while the mass of the displaced water is $m_w = \frac{1}{3}\rho_w V$. Therefore, the mass of the cylinder is

$$m = \frac{1}{3}(\rho_o + \rho_w)V.$$

In the second part of the experiment, when the cylinder floats in oil only, it must displace the mass of oil $m_o' = m$, which will have the volume

$$V_o' = \frac{m_o'}{\rho_o} = \frac{m}{\rho_o} = \frac{\frac{1}{3}(\rho_o + \rho_w)V}{\rho_o}.$$

The fraction of the cylinder that is submerged in oil is then

$$\frac{V_o'}{V} = \frac{\frac{1}{3}(\rho_o + \rho_w)}{\rho_o} = \frac{1}{3}\left(1 + \frac{\rho_w}{\rho_o}\right) = \frac{3}{4} \quad \Rightarrow \quad \text{the answer is B.}$$

2016.3. Let us introduce the following notation: $H = 3$ m, $h = 1$ m, and $m = 5$ kg. Let the positive direction be vertically down and let v be the speed of the book when it hits the snow. Then

$$v^2 = 0 + 2gH.$$

That same speed is reduced to 0 due to a, the acceleration of snow. So, for the book penetrating the snow, we have $0 = v^2 + 2ah$, therefore

$$2gH = v^2 = -2ah \quad \Rightarrow \quad a = -\frac{H}{h}g \quad \text{as expected, } a < 0.$$

Now, what are the forces acting on the book? In addition to the force of snow F_s acting upwards, there is also the weight of the book, mg, acting downwards, therefore Newton's Second Law is $F_{net} = mg - F_s = ma$, therefore

$$F_s = m(g - a) = m\left(g + \frac{H}{h}g\right) = mg\left(1 + \frac{H}{h}\right) = 4mg = 200 \text{ N},$$

so the answer is E.

2016.4. The bead in this problem moves in 3D, on a helical wire. Let the radius of the helix be r and let the angle between a tangent to the helix and the horizontal plane be α. The acceleration of the bead in the direction tangential to the wire is $a_w = g\sin\alpha$. However, that is not the total acceleration. The vector of total acceleration also has the centripetal (radial) component, $a_c = \frac{v_h^2}{r}$, where v_h is the horizontal component of the bead's velocity. Note that $\vec{a}_c$ is perpendicular to $\vec{a}_w$ and changes with time, because the bead speeds up.

The analysis so far allows us to eliminate answers A, B, and C (because we know a is not constant over time, so A is out, and because we know at $t = 0$ the acceleration is not zero, so B and C are out). In the competition that may be all we can do in the given time. To determine exactly what happens, let us consider Figure 1 and note that $\vec{a}_w$ can be decomposed into two orthogonal components: one vertically down,

$$a_v = a_w \sin\alpha = g\sin^2\alpha,$$

and the other always horizontal and tangential to the cylinder that envelopes the helix,

$$a_h = a_w \cos\alpha = g\sin\alpha\cos\alpha.$$

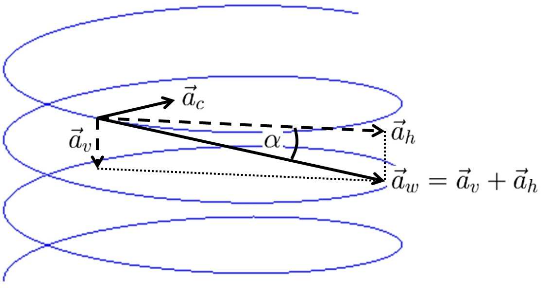


Figure 1: Illustration for Problem 2016.4. The vector of total acceleration $\vec{a}$ (not shown in the figure) has perpendicular components $\vec{a}_w$ and $\vec{a}_c$, i.e., $\vec{a} = \vec{a}_w + \vec{a}_c$. Furthermore, $\vec{a}_w$ has perpendicular components $\vec{a}_v$ and $\vec{a}_h$.

The magnitude of total acceleration vector is

$$a = \sqrt{a_w^2 + a_c^2},$$

where

$$a_w = g \sin\alpha \qquad \text{and} \qquad a_c = \frac{v_h^2}{r} = \frac{(a_h t)^2}{r} = \frac{(gt \sin\alpha \cos\alpha)^2}{r}.$$

Finally,

$$a = \sqrt{a_w^2 + a_c^2} = \sqrt{P + Qt^4},$$

where P and Q are positive constants. We see that for sufficiently small values of t, this function has a positive value independent of t. For sufficiently large values of t, this function behaves like a quadratic function. The only answer that looks like that is D.

2016.5. Note that in this problem the box is accelerating to the right only; it is not in free fall. In the "world" reference system, the trajectory of the particle is a typical projectile trajectory, a parabola, so A and B cannot be answers (in A the particle ceases to fall, while in B it flies up). If the box did not accelerate to the right, the answer would have been E, but with box acceleration to the right, the answer is either C or D. However, the end behavior in D is as if the box is accelerating to the left instead of right. Alternatively, note that since uniform acceleration is equivalent to gravity, the effective gravity is pointed to the bottom left and so is the parabola. Either way, the answer is C.

2016.6. Among given options, the only graph that can be obtained as a special case of a parabola (as in Problem 2016.5) is the line in answer A. Furthermore, B does not match the initial angle of the velocity vector, while answers C, D, and E all include upward motion of the particle, against gravity, so they are all out except A. Thus, the answer is A.

2016.7. This is a very hard problem, so in the competition you probably want to skip it (or, if there is no penalty for wrong answers, make a guess). Nevertheless, it is a very interesting problem, so let's look at it. Let the relaxed spring length and the spring constant of each half of the spring be L and k, respectively. For perpendicular extension A, each half of the spring becomes $L+x$ long (we will determine x from Pythagorean theorem) and the potential energy of the whole system is

$$E_p = 2 \cdot \frac{1}{2} k x^2 = k x^2.$$

From Pythagorean theorem we have $(L+x)^2 = L^2 + A^2$, so we find

$$x = \sqrt{L^2 + A^2} - L = L\left(\sqrt{1 + \frac{A^2}{L^2}} - 1\right)$$

Two extreme cases are of interest here:

- $A \gg L \quad \Rightarrow \quad x \approx A \quad \Rightarrow \quad E_p \approx kA^2$

  This quadratic relationship tells us that for $A \gg L$ we have harmonic oscillations, so the period of oscillations does not depend on A.

- $A \ll L \quad \Rightarrow \quad x \approx L(1 + \frac{1}{2}\frac{A^2}{L^2} - 1) = \frac{A^2}{2L} \quad \Rightarrow \quad E_p \approx \frac{k}{4L^2} A^4$

  Here we used the approximation[1] $\sqrt{1+h} \approx 1 + \frac{1}{2}h$, which is valid when $h \ll 1$. The quartic dependence of E_p on A tells us that we have non-harmonic oscillations and the period of oscillations will depend on A.

With the analysis so far we have eliminated answers A, D, and E. To see whether for small values of A the period increases or decreases as A increases[2], we recall that force on the mass can be expressed as the negative derivative[3] of the potential energy[4]. In our case $F(A) = -\frac{d}{dA}E_p(A) \approx -\frac{d}{dA}\left(\frac{k}{4L^2}A^4\right) = -\frac{k}{L^2}A^3$. We see that the magnitude of force is not a linear function of the displacement A and in fact it grows faster than A, implying that for small values of A the period of oscillation *decreases* with increasing A, so the correct answer is C.

[1] A more general formula valid for $h \ll 1$ and any real a is $(1+h)^a \approx 1 + ah$.

[2] Qualitatively, the answer can be found without calculus. Note that the smaller A gets, A^4 is getting progressively smaller than A^2. At the same time, the restoring force is getting smaller and smaller than what is required for a simple harmonic oscillator. This in turn decreases acceleration and increases the period. Thus, we can say that for small A, the period decreases as A increases.

[3] Did someone say that $F = ma$ contest does not require calculus? Well, for the most part it does not, but the hardest problems sometimes do require it.

[4] For example, for a harmonic spring $F = -kx$, $E_p = \frac{1}{2}kx^2$, and indeed $F(x) = -\frac{d}{dx}E_p(x)$.

2016.8. Newton showed that Kepler's First and Third Laws are consequences of the inverse-square ($1/r^2$) nature of gravitation. Kepler's Second Law follows from the law of conservation of angular momentum and is independent of the nature of the central force. The correct answer is B.

2016.9. Let us denote the mass of the bead by m, the angle at which the bead first loses contact with the sphere by θ, and the corresponding height difference by h (see Figure 2). We can write the following equations:

$$mgh = \frac{mv^2}{2} \qquad \text{(conservation of energy)}$$

$$h = R(1-\cos\theta) \qquad \text{(geometry of the problem)}$$

$$mg\cos\theta \geq \frac{mv^2}{R} \qquad \text{(condition on forces for contact).}$$

Eliminate $\cos\theta = \frac{v^2}{gR}$ to get $v = \sqrt{\frac{2gR}{3}} = \sqrt{\frac{gD}{3}}$, so the answer is E.

Note: Going back to $\cos\theta = \frac{v^2}{gR}$, it turns out that $\cos\theta = \frac{2}{3} \quad \Rightarrow \quad \theta = 48.2^o$.

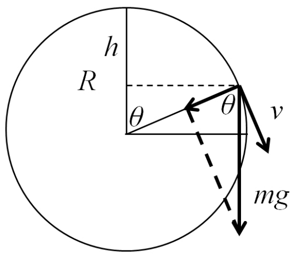


Figure 2: Illustration for Problem 2016.9.

2016.10. In Figure 3 we draw free body diagrams for bodies 1 (top) and 2 (bottom):

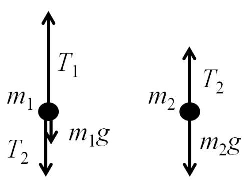


Figure 3: Illustration for Problem 2016.10.

Before the top string was cut, forces on each body were balanced, so

$$T_1 = T_2 + m_1 g \qquad \text{and} \qquad T_2 = m_2 g,$$

therefore $T_1 = (m_1 + m_2)g$.

Immediately after the top string is cut, the only change in the free body diagrams is that T_1 is not there any more, so the net force on body 1 is

$$F_1 = T_2 + m_1 g = (m_1 + m_2)g \quad \text{downwards},$$

while the net force on body 2 remains, at least immediately after the cut,[5]

$$F_2 = 0.$$

Therefore, the accelerations immediately after the cut are

$$a_1 = \frac{F_1}{m_1} = \left(1 + \frac{m_2}{m_1}\right) g = 30 \text{ N} \qquad \text{and} \qquad a_2 = \frac{F_2}{m_2} = 0,$$

so the correct answer is D.

2016.11. Let the side of the cube be L. Since the density is constant, the mass of the car, m, is proportional to its volume, V, i.e.,

$$m \propto V = L^3 \qquad \Rightarrow \qquad L \propto m^{1/3}.$$

Therefore, for the cross-section A we have

$$A = L^2 \propto m^{2/3}.$$

The speed of the car reaches its maximum value v_{max} and stays there (so it is constant), when the force F generated by the engine[6] equals the air friction in magnitude. With that, the formula relating power, force, and speed, $P = Fv$, becomes

$$P = Fv_{max} = F_s v_{max} \propto Av_{max}^2 v_{max} \propto m^{2/3} v_{max}^3.$$

But $P \propto m$, so we can write

$$m \propto m^{2/3} v_{max}^3 \qquad \Rightarrow \qquad v_{max} \propto m^{1/9},$$

so the correct answer is A.

[5]It is important to emphasize that this is true only immediately after the top string is cut. As the top body begins to fall, T_2 diminishes, and soon the forces on the two bodies will be just their weights.

[6]What propels a car? The answer involves friction (for this purpose, static friction is better than kinetic friction) between the wheels and the ground as a positive contributor to motion!

2016.12. According to Archimedes' Principle, before the container is accelerated, the fraction of the solid's volume submerged in liquid is $x = \rho/\rho_w$. Why? The object floats, so the weight of the (entire) object equals the force of buoyancy, i.e., the weight of the displaced fluid:

$$mg = m_w g \qquad \Rightarrow \qquad \rho V g = \rho_w x V g \qquad \Rightarrow \qquad x = \rho/\rho_w.$$

We see that x does not depend on g, so the submerged fraction does not change during acceleration, and the answer is C. Now, you might ask, what does g have to do with kinematic acceleration? Just recall Einstein's thought experiments with elevators, where he realized g is indiscernible from vertical acceleration a. That is his famous Equivalence Principle. Another way to explain this result is to say that in the reference system of the container and the elevator that is accelerating it say up, the solid becomes heavier, its weight is now $m(g + a)$, but so does the displaced liquid, it now weighs $m_w(g + a)$. If we use that as the starting point of the above derivation, we again get $x = \rho/\rho_w$ and the answer is C.

2016.13. This problem is almost identical to Problem 2014.16. Let us make a slight change in the notation for this problem in order to be able to explore what happens in a more general case:

- Let the velocities of the two objects before the collision be v_1 and v_2.
- Let the velocities of the two objects after the collision be w_1 and w_2.
- Then the ratio r_2 is defined as $r_2 = \left|\frac{w_2}{v_1}\right|$.

One extreme case of collisions are *elastic collisions* (think air hockey pucks), for which both linear momentum and kinetic energy are conserved so

$$m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2 \quad \text{and} \quad \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} = \frac{m_1w_1^2}{2} + \frac{m_2w_2^2}{2}$$

These equations imply $v_1 + w_1 = v_2 + w_2$ (a useful formula in its own right), and finally

$$w_1 = \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \qquad \text{and} \qquad w_2 = \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2}.$$

The other extreme case of collisions are *perfectly inelastic collisions* (think two ice cream scoops hitting each other and merging in the process). In that case only the linear momentum is conserved

$$m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2$$

and objects get merged, so $w_1 = w_2 = w$.
In that case

$$w = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}.$$

Back to our problem, with $v_2 = 0$, we find that for elastic collisions

$$r_2 = \left|\frac{w_2}{v_1}\right| = \frac{2m_1}{m_1 + m_2} = \frac{2}{1+\alpha},$$

while for perfectly inelastic collisions we have

$$r_2 = \left|\frac{w_2}{v_1}\right| = \frac{m_1}{m_1 + m_2} = \frac{1}{1+\alpha}.$$

In this sense, all other collisions (partially inelastic collisions) lie between these two extremes[7] and in general

$$\frac{1}{1+\alpha} \leq r_2 \leq \frac{2}{1+\alpha}, \qquad \text{which is answer E.}$$

Note: It is not a bad idea to memorize the formulas for w_1, w_2, and w that we (almost) derived above, at least for the day of the competition.

2016.14. In this problem we can avoid using calculus if we know the following trick to derive period of oscillations: If displacement x causes acceleration a, then the period of oscillations is

$$T = 2\pi\sqrt{\frac{x}{a}}.$$

In our problem the displacement (spring extension) x causes $a = \alpha r$, where initially $r = \ell$ and later $r = \ell/2$. Euler's Second Law (the rotational analog of Newton's Second Law) tells us that

$$I\alpha = \tau = kxr \quad \Rightarrow \quad \alpha = \frac{kxr}{I} \quad \Rightarrow \quad a = \alpha r = \frac{kxr^2}{I},$$

therefore

$$T = 2\pi\sqrt{\frac{x}{a}} = 2\pi\sqrt{\frac{I}{kr^2}} \propto \frac{1}{r} \quad \Rightarrow \quad f = \frac{1}{T} \propto r.$$

This tells us that the new frequency, after we change from $r = \ell$ to $r = \ell/2$, is $f' = f/2$, and the answer is A.

2016.15. Again, we use the following trick: If displacement x causes acceleration a, then the period of oscillations is

$$T = 2\pi\sqrt{\frac{x}{a}}.$$

[7] See a footnote about this in Problem 2014.16.

As shown in Figure 4, the displacement is $x = \ell \sin\theta \approx \ell\theta$. It corresponds to $a = \alpha\ell$. From

$$I\alpha = \tau = (F\sin\theta)\ell \approx (k\ell\theta)\ell = kx\ell$$

we find

$$\alpha = \frac{kx\ell}{I} \qquad \Rightarrow \qquad a = \alpha\ell = \frac{kx\ell^2}{I},$$

yielding the same frequency as for $r = \ell$ in Problem 2016.14. The answer is C.

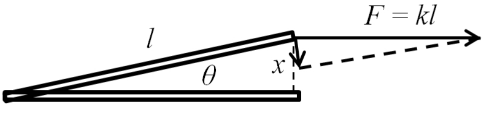


Figure 4: Illustration for Problem 2016.15.

2016.16. Let $v = 10$ m/s, $u = 8$ m/s, and $h = 1.25$ m. The vertical displacement due to free fall is $h = \frac{gt^2}{2}$, so $t = \sqrt{2h/g}$. During that time the ball also moves horizontally, $s = ut$ behind the truck, i.e.,

$$s = ut = u\sqrt{2h/g} = 4 \text{ m} \quad \Rightarrow \quad \text{the answer is C.}$$

Note: The problem is asked relative to the truck, so v does not matter.

2016.17. Initially, when the ball touches the ground, its linear and angular momentum are

$$p = mv \qquad \text{and} \qquad L = I\omega = \frac{2}{3}mR^2\omega.$$

Due to friction, both of these quantities diminish over contact time t according to the following laws:

$$Ft = m\Delta v \qquad \text{and} \qquad FRt = \frac{2}{3}mR^2\Delta\omega.$$

For a given ω, the critical speed $v = v_c$ for which both translational and rotational motion stop at the same time t, can be found from

$$Ft = mv_c \quad \text{and} \quad FRt = \frac{2}{3}mR^2\omega \quad \Rightarrow \quad v_c = \frac{2}{3}R\omega \quad \Rightarrow \quad \text{the answer is A.}$$

Note: If the force of friction F is given, or at least the coefficient of kinetic friction μ_k (so then $F = \mu_k mg$), the time over which this happens is

$$t = \frac{mv_c}{F} = \frac{2mR\omega}{3F} = \frac{2R\omega}{3\mu_k g}.$$

If we plug in reasonable values of $R = 0.02$ m, $\mu_k = 0.5$, and $\omega = 1$ rad/s, we get $t = 2.7$ ms.

2016.18. The total angle is 3 full rotations, so (see Figure 5):

$$\theta = 6\pi \text{ rad} = \frac{1}{2}\alpha t_1^2 + \omega t_2 + \frac{1}{2}\alpha t_3^2.$$

The angular acceleration during the speed-up and the slow-down has the same magnitude so

$$t_1 = t_3 = t = \frac{\omega}{\alpha} = 5 \text{ s}.$$

From this

$$t_2 = \frac{\theta - \frac{1}{2}\alpha t_1^2 - \frac{1}{2}\alpha t_3^2}{\omega} = \frac{\theta - \alpha t^2}{\omega} = \frac{\theta - \frac{\omega^2}{\alpha}}{\omega} = \frac{\theta}{\omega} - \frac{\omega}{\alpha} = 85 \text{ s}.$$

Finally,

$$T = t_1 + t_2 + t_3 = \frac{\theta}{\omega} + \frac{\omega}{\alpha} = 95 \text{ s} \quad \Rightarrow \quad \text{the answer is E.}$$

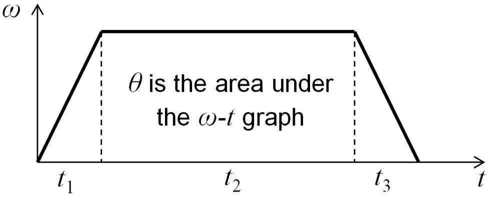


Figure 5: Illustration for Problem 2016.18.

2016.19. We can determine the speed v of the bead as it leaves the wire by using the conservation of energy:

$$\frac{mv^2}{2} = mg(2R) \qquad \Rightarrow \qquad v^2 = 4gR.$$

For the vertical component of projectile motion we can write

$$H = \frac{gt^2}{2} \qquad \Rightarrow \qquad t^2 = \frac{2H}{g},$$

while for the horizontal component we have

$$D = vt \qquad \Rightarrow \qquad D^2 = v^2 t^2 = 4gR\frac{2H}{g} = 8RH.$$

Therefore

$$RH = \frac{1}{8}D^2.$$

We see that the relationship between RH and D^2 is purely linear, i.e., with zero intercept, therefore the answer is D.

2016.20. Sliding begins when $\tan\theta = \mu$. Why? Recall that the maximum static friction is

$$F_s^{max} = \mu F_n = \mu mg\cos\theta,$$

and so we need the "parallel" component $mg\sin\theta$ of the weight mg to be

$$mg\sin\theta = F_s^{max} = \mu mg\cos\theta \qquad \Rightarrow \qquad \mu = \tan\theta.$$

From Figure 6 we see that

$$\mu = \tan\theta = \frac{h}{h/3} = 3 \quad \Rightarrow \quad \text{so the answer is E.}$$

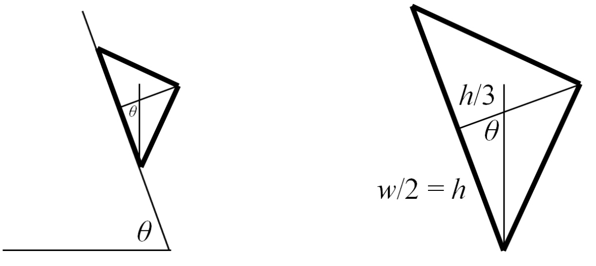


Figure 6: Illustration for Problem 2016.20. Toppling happens when the centroid (center of mass for uniform objects) is above the lowest corner of the prism (recall that the centroid divides the altitude in ratio 1:2).

2016.21. There are no horizontal forces on the system, therefore its center of mass will not move horizontally. Because the ratio of the masses is 3 : 1, we see that the lighter ball will move horizontally by $x = \frac{3}{3+1}L = \frac{3}{4}L$. The answer is A.

2016.22. Initially, there was only potential energy $E_p = mgL$. At the very end, the heavier ball is not moving[8] while the lighter ball is hitting the ground with vertical velocity v. From conservation of energy

$$\frac{mv^2}{2} = mgL \qquad \Rightarrow \qquad v = \sqrt{2gL} \qquad \Rightarrow \qquad \text{the answer is A.}$$

[8] This is because at any time, the center of mass has 0 horizontal velocity and at the end of the fall, the rod is horizontal, therefore the balls will also have 0 horizontal velocity.

2016.23. Divide the ring into n small pieces as in Figure 7. For each piece:

$$m = M/n \quad \text{and} \quad \theta = 2\pi/n.$$

The force of tension towards the piece to the left is

$$T = k\Delta x = 2\pi k(R' - R).$$

Its radial component is half of the centripetal force (the other half comes from the tension force towards the piece to the right), so

$$2T \sin\frac{\theta}{2} = m\omega^2 R'.$$

For large n the angle θ is small, so we can use the approximation[9] $\sin\frac{\theta}{2} \approx \frac{\theta}{2}$

$$2T\frac{\theta}{2} = m\omega^2 R' \qquad \Rightarrow \qquad T\frac{2\pi}{n} = \frac{M}{n}\omega^2 R'$$

Finally, from $T = 2\pi k(R' - R)$,

$$4\pi^2 k(R' - R) = M\omega^2 R' \quad \Rightarrow \quad R' = \frac{4\pi^2 kR}{4\pi^2 k - M\omega^2} \quad \Rightarrow \quad \text{the answer is D.}$$

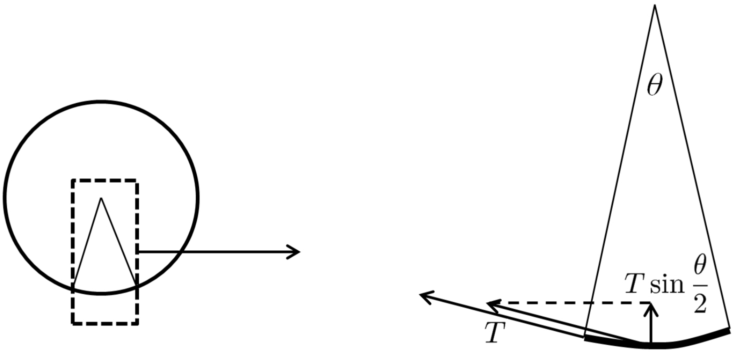


Figure 7: Illustration for Problem 2016.23.

[9]For angles $x \ll 1$ rad we have

$$\sin x \approx x - \frac{x^3}{6} \approx x \qquad \cos x \approx 1 - \frac{x^2}{2} \approx 1 \qquad \tan x \approx x - \frac{x^3}{3} \approx x.$$

Note that these approximations must be modified if angle x is not expressed in radians.

2016.24. We are given the moment of inertia of an equilateral triangle around one of its sides,

$$I_s = \frac{1}{8}ma^2.$$

From the parallel axis theorem, the moment of inertia for the axis parallel to a side and containing the centroid (note that the centroid divides the height in ratio 1:2 and that the height is $h = \frac{a\sqrt{3}}{2}$) is

$$I_c = I_s - \left(\frac{1}{3}h\right)^2 = \frac{1}{8}ma^2 - m\left(\frac{1}{3}\frac{a\sqrt{3}}{2}\right)^2 = \frac{1}{24}ma^2.$$

Again from the parallel axis theorem, the moment of inertia for the axis parallel to a side and containing the opposite vertex is

$$I_v = I_c + \left(\frac{2}{3}h\right)^2 = \frac{1}{24}ma^2 + m\left(\frac{2}{3}a\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{8}ma^2.$$

A regular hexagon with mass m can be divided into six equilateral triangles, each with mass $m/6$. Four of them have the moment of inertia given by $\frac{1}{8}\cdot\frac{m}{6}a^2$, while the moments of inertia of the remaining two are $\frac{3}{8}\cdot\frac{m}{6}a^2$. The total moment of inertia is then

$$I_6 = 4\cdot\frac{1}{8}\cdot\frac{m}{6}a^2 + 2\cdot\frac{3}{8}\cdot\frac{m}{6}a^2 = \frac{5}{24}ma^2 \quad\Rightarrow\quad \text{the answer is B.}$$

2016.25. This problem does not say it explicitly, but we usually assume that the measurement errors have Gaussian distribution. When Gaussian measurements are added or subtracted, their measurement uncertainties (given by their standard deviations) are added using the geometric (Pythagorean-like) sum[10]. Therefore:

- The uncertainty for Alice's measurement is $\Delta A = 2$ mm
- For Bob it is $\Delta B = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ mm
- For Christina it is $\Delta C = \sqrt{1^2 + 1^2} = \sqrt{2}$ mm

Therefore,

$$\Delta C < \Delta A < \Delta B \quad\Rightarrow\quad \text{the answer is A.}$$

[10] To read more about the background theory for this problem, go to `http://ipl.physics.harvard.edu/wp-uploads/2013/03/PS3_Error_Propagation_sp13.pdf` or search the Web for "error analysis" or other related topics.

Year 2017

$\boldsymbol{F = ma}$ Exam

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2017

2017 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2017.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

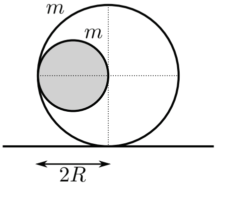
1. A motorcycle rides on the vertical walls around the perimeter of a large circular room. The friction coefficient between the motorcycle tires and the walls is μ. How does the minimum μ needed to prevent the motorcycle from slipping downwards change with the motorcycle's speed, s?

   (A) $\mu \propto s^0$
   (B) $\mu \propto s^{-1/2}$
   (C) $\mu \propto s^{-1}$
   (D) $\mu \propto s^{-2}$
   (E) none of these

2. A mass m hangs from a massless spring connected to the roof of a box of mass M. When the box is held stationary, the mass-spring system oscillates vertically with angular frequency ω. If the box is dropped falls freely under gravity, how will the angular frequency change?

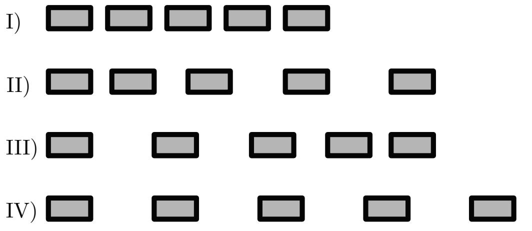
   (A) ω will be unchanged
   (B) ω will increase
   (C) ω will decrease
   (D) Oscillations are impossible under these conditions
   (E) ω will increase or decrease depending on the values of M and m

3. A ball of radius R and mass m is magically put inside a thin shell of the same mass and radius $2R$. The system is at rest on a horizontal frictionless surface initially. When the ball is, again magically, released inside the shell, it sloshes around in the shell and eventually stops at the bottom of the shell. How far does the shell move from its initial contact point with the surface?



   (A) R.
   (B) $R/2$.
   (C) $R/4$.
   (D) $3R/8$.
   (E) $R/8$

4. Several identical cars are standing at a red light on a one-lane road, one behind the other, with negligible (and equal) distance between adjacent cars. When the green light comes up, the first car takes off to the *right* with constant acceleration. The driver in the second car reacts and does the same 0.2 s later. The third driver starts moving 0.2 s after the second one and so on. All cars accelerate until they reach the speed limit of 45 km/hr, after which they move to the right at a constant speed. Consider the following patterns of cars.



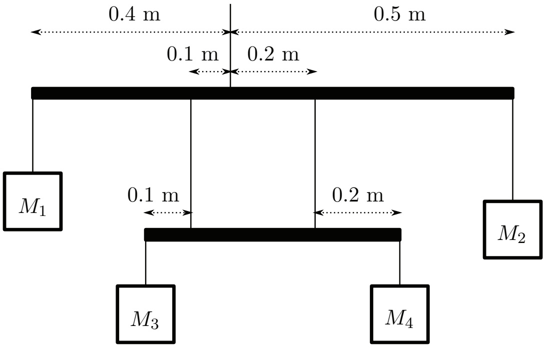
Just before the first car starts accelerating to the right, the car pattern will qualitatively look like the pattern in I. After that, the pattern will qualitatively evolve according to

(A) First I, then II, and then III.
(B) First I, then II, and then IV.
(C) First I, and then IV, with neither II nor III as intermediate stage.
(D) First I, and then II.
(E) First I, and then III.

5. A projectile is launched with speed v_0 off the edge of a cliff of height h, at an angle θ from the horizontal. Air friction is negligible. To maximize the horizontal range of the projectile, θ should satisfy

(A) $45^\circ < \theta < 90^\circ$
(B) $\theta = 45^\circ$
(C) $0^\circ < \theta < 45^\circ$
(D) $\theta = 0^\circ$
(E) $\theta < 45^\circ$ or $\theta > 45^\circ$, depending on the values of h and v_0.

6. In the mobile below, the two cross beams and the seven supporting strings are all massless. The hanging objects are $M_1 = 400$ g, $M_2 = 200$ g, and $M_4 = 500$ g. What is the value of M_3 for the system to be in static equilibrium?



   (A) 300 g
   (B) 400 g
   (C) 500 g
   (D) 600 g
   (E) 700 g

The following information applies to questions 7 and 8.

A train, originally of mass M, is traveling on a frictionless straight horizontal track with constant speed v. Snow starts to fall vertically and sticks to the train at a rate of ρ, where ρ has units of kilograms per second. The train's engine keeps the train moving at constant speed v as snow accumulates on the train.

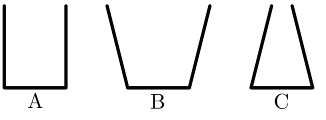
7. The rate at which the kinetic energy of the train and snow increases is

   (A) 0
   (B) Mgv
   (C) $\frac{1}{2}Mv^2$
   (D) $\frac{1}{2}\rho v^2$
   (E) ρv^2

8. The minimum power required from the engine to keep the train traveling at a constant speed v is

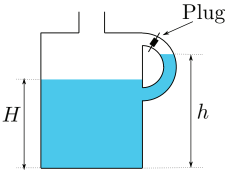
   (A) 0
   (B) Mgv
   (C) $\frac{1}{2}Mv^2$
   (D) $\frac{1}{2}\rho v^2$
   (E) ρv^2

9. Flasks A, B, and C each have a circular base with a radius of 2 cm. An equal volume of water is poured into each flask, and none overflow. Rank the force of water F on the base of the flask from greatest to least.



(A) $F_A > F_B > F_C$
(B) $F_A > F_C > F_B$
(C) $F_B > F_C > F_A$
(D) $F_C > F_A > F_B$
(E) $F_A = F_B = F_C$

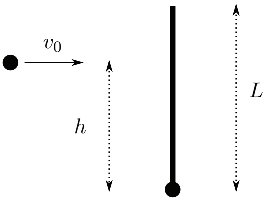
10. The handle of a gallon of milk is plugged by a manufacturing defect. After removing the cap and pouring out some milk, the level of milk in the main part of the jug is lower than in the handle, as shown in the figure. Which statement is true of the gauge pressure P of the milk at the bottom of the jug? ρ is the density of the milk.



(A) $P = \rho g h$
(B) $P = \rho g H$
(C) $\rho g H < P < \rho g h$
(D) $P > \rho g h$
(E) $P < \rho g H$

The following information applies to questions 11 and 12.

A small hard solid sphere of mass m and negligible radius is connected to a thin rod of length L and mass $2m$. A second small hard solid sphere, of mass M and negligible radius, is fired perpendicularly at the rod at a distance h above the sphere attached to the rod, and sticks to it.



11. In order for the rod not to rotate after the collision, the second sphere should hit the thin rod at

    (A) $h = 0$
    (B) $h = L/3$
    (C) $h = L/2$
    (D) $h = L$
    (E) Any location L will work

12. In order for the rod not to rotate after the collision, the second sphere should have a mass M given by

    (A) $M = m$
    (B) $M = 1.5m$
    (C) $M = 2m$
    (D) $M = 3m$
    (E) Any mass M will work.

13. A massless rope passes over a frictionless pulley. Particles of mass M and $M + m$ are suspended from the two different ends of the rope. If $m = 0$, the tension T in the pulley rope is Mg. If instead the value m increases to infinity, the value of the tension

    (A) stays constant
    (B) decreases, approaching a nonzero constant
    (C) decreases, approaching zero
    (D) increases, approaching a finite constant
    (E) increases to infinity

The following information applies to questions 14 and 15.

An object starting from rest can roll without slipping down an incline.

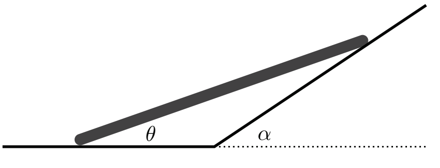
14. Which of the following four objects, each with mass M and radius R, would have the largest acceleration down the incline?

    (A) A uniform solid sphere
    (B) A uniform solid disk
    (C) A hollow spherical shell
    (D) A hoop
    (E) All objects would have the same acceleration.

15. Which of the following four objects, each a uniform solid sphere released from rest, would have the largest speed after the center of mass has moved through a vertical distance h?

    (A) A sphere of mass M and radius R.
    (B) A sphere of mass $2M$ and radius $\frac{1}{2}R$.
    (C) A sphere of mass $M/2$ and radius $2R$.
    (D) A sphere of mass $3M$ and radius $3R$.
    (E) All objects would have the same speed.

16. A rod moves freely between the horizontal floor and the slanted wall. When the end in contact with the floor is moving at v, what is the speed of the end in contact with the wall?

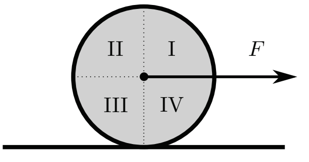


    (A) $v\dfrac{\sin\theta}{\cos(\alpha-\theta)}$.
    (B) $v\dfrac{\sin(\alpha-\theta)}{\cos(\alpha+\theta)}$.
    (C) $v\dfrac{\cos(\alpha-\theta)}{\sin(\alpha+\theta)}$.
    (D) $v\dfrac{\cos\theta}{\cos(\alpha-\theta)}$.
    (E) $v\dfrac{\sin\theta}{\cos(\alpha+\theta)}$.

17. An object is thrown directly downward from the top of a 180 meter tall building. It takes 1.0 seconds for the object to fall the last 60 meters. With what initial downward speed was the object thrown from the roof?

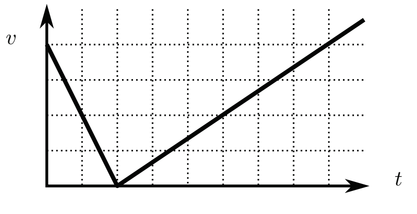
(A) 15 m/s
(B) 25 m/s
(C) 35 m/s
(D) 55 m/s
(E) insufficient information.

18. A uniform disk is being pulled by a force F through a string attached to its center of mass. Assume that the disk is rolling smoothly without slipping. At a certain instant of time, in which region of the disk (if any) is there a point moving with zero total acceleration?



(A) Region I
(B) Region II
(C) Region III
(D) Region IV
(E) All points on the disk have non-zero acceleration.

19. A puck is kicked up a ramp, which makes an angle of 30° with the horizontal. The graph below depicts the speed of the puck versus time. What is the coefficient of friction between the puck and the ramp?



(A) 0.07
(B) 0.15
(C) 0.22
(D) 0.29
(E) 0.37

The following information applies to questions 20, 21, and 22.

A particle of mass m moving at speed v_0 collides with a particle of mass M which is originally at rest. The fractional momentum transfer f is the absolute value of the final momentum of M divided by the initial momentum of m.

20. If the collision is completely *inelastic*, under what condition will the fractional momentum transfer between the two objects be a maximum?

    (A) $m/M \ll 1$
    (B) $0.5 < m/M < 1$
    (C) $m = M$
    (D) $1 < m/M < 2$
    (E) $m/M \gg 1$

21. If the collision is perfectly *elastic*, what is the maximum possible fractional momentum transfer, $f_{\max}$?

    (A) $0 < f_{\max} < \frac{1}{2}$
    (B) $f_{\max} = \frac{1}{2}$
    (C) $\frac{1}{2} < f_{\max} < \frac{3}{2}$
    (D) $f_{\max} = 2$
    (E) $f_{\max} \geq 3$

22. The fractional energy transfer is the absolute value of the final kinetic energy of M divided by the initial kinetic energy of m.

    If the collision is perfectly elastic, under what condition will the fractional energy transfer between the two objects be a maximum?

    (A) $m/M \ll 1$
    (B) $0.5 < m/M < 1$
    (C) $m = M$
    (D) $1 < m/M < 2$
    (E) $m/M \gg 1$

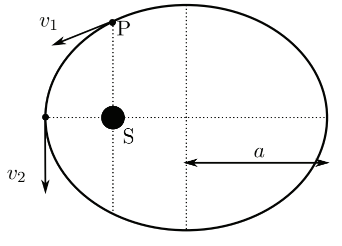
23. A spring has a length of 1.0 meter when there is no tension on it. The spring is then stretched between two points 10 meters apart. A wave pulse travels between the two end points in the spring in a time of 1.0 seconds. The spring is now stretched between two points that are 20 meters apart. The new time it takes for a wave pulse to travel between the ends of the spring is closest to

    (A) 0.5 seconds
    (B) 0.7 seconds
    (C) 1 second
    (D) 1.4 seconds
    (E) 2 seconds

24. A ball of mass m moving at speed v collides with a massless spring of spring constant k mounted on a stationary box of mass M in free space. No mechanical energy is lost in the collision. If the system does not rotate, what is the maximum compression x of the spring?

 (A) $x = v\sqrt{\frac{mM}{(m+M)k}}$
 (B) $x = v\sqrt{\frac{m}{k}}$
 (C) $x = v\sqrt{\frac{M}{k}}$
 (D) $x = v\sqrt{\frac{m+M}{k}}$
 (E) $x = v\sqrt{\frac{(m+M)^3}{mMk}}$

25. A planet orbits around a star S, as shown in the figure. The semi-major axis of the orbit is a. The perigee, namely the shortest distance between the planet and the star is $0.5a$. When the planet passes point P (on the line through the star and perpendicular to the major axis), its speed is v_1. What is its speed v_2 when it passes the perigee?



 (A) $v_2 = \frac{3}{\sqrt{5}}v_1$.
 (B) $v_2 = \frac{3}{\sqrt{7}}v_1$.
 (C) $v_2 = \frac{2}{\sqrt{3}}v_1$.
 (D) $v_2 = \frac{\sqrt{7}}{\sqrt{3}}v_1$.
 (E) $v_2 = 4v_1$.

Answers, Problem Difficulty, and Topics

ToC

2017.1	D	⋆	static friction, centripetal force
2017.2	B	⋆ ⋆	springs, oscillations
2017.3	B	⋆	center of mass
2017.4	B	⋆ ⋆	linear motion
2017.5	C	⋆ ⋆ ⋆	projectile motion
2017.6	E	⋆ ⋆	forces, torque, equilibrium
2017.7	D	⋆	kinetic energy
2017.8	E	⋆ ⋆	linear momentum
2017.9	D	⋆ ⋆	pressure
2017.10	B	⋆ ⋆	pressure
2017.11	B	⋆	center of mass
2017.12	E	⋆	torque
2017.13	D	⋆ ⋆	forces, Atwood machine
2017.14	A	⋆	rolling motion, moment of inertia
2017.15	E	⋆	rolling motion, moment of inertia
2017.16	D	⋆ ⋆ ⋆	linear motion
2017.17	B	⋆	linear motion
2017.18	D	⋆ ⋆ ⋆	circular motion
2017.19	D	⋆ ⋆	kinetic friction
2017.20	A	⋆ ⋆	collisions
2017.21	D	⋆ ⋆	collisions
2017.22	C	⋆ ⋆	collisions
2017.23	C	⋆	waves
2017.24	A	⋆ ⋆	springs, collisions, energy
2017.25	A	⋆ ⋆ ⋆	Kepler's Laws

Solutions

ToC

2017.1. The wall's static friction is the force preventing the motorcyclist from slipping downward. Let m be the total mass of the motorcycle and cyclist. The normal force N on the motorcyclist provides a constant centripetal acceleration and can be calculated with the formula

$$N = ma_c = \frac{mv^2}{r}.$$

The motorcyclist does not slip downward only when its weight is less than the maximum force of static friction μN, so

$$mg < \mu N = \mu \frac{mv^2}{r} \quad \Rightarrow \quad \mu > \frac{gr}{v^2}.$$

Then the minimum value of μ is proportional to v^{-2}, so going back to the original problem notation, the answer is D.

2017.2. A mass-spring system with a constant external force applied to it (such as gravity) will have the same frequency of oscillation; only the equilibrium length will be changed. Because of this, it is easy to think at first glance that the spring's frequency of oscillation does not change in this problem.

However, this is incorrect because the box is also part of the system. When the box is held stationary, the problem is a typical mass-spring system with one mass and one fixed end. However, when it is dropped, the box is no longer held stationary, so the system turns into a spring with two masses, m and M, on either end. Letting x be the displacement of the spring from its equilibrium length, the acceleration of the mass in the box's reference frame is then

$$a = -\frac{F}{m} - \frac{F}{M} = -kx\left(\frac{1}{m} + \frac{1}{M}\right).$$

If we compare this to the original equation of motion, the angular frequency of the spring increases from $\sqrt{k/m}$ to $\sqrt{k/m + k/M}$. Thus, the answer is B.

2017.3. Since the system begins at rest, and there is zero net external force on the system in the horizontal direction, the x-coordinate of its center of mass must remain constant by Newton's First Law. Furthermore, when the ball eventually reaches its final position at the bottom, the contact point of the shell with the surface is vertically aligned with the center of mass due to symmetry. If we let the x-coordinate of the initial contact point be $x_i = 0$, the center of mass can be found by averaging that of the ball and the shell, yielding

$$x_f = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m \cdot 0 + m \cdot R}{m + m} = \frac{R}{2} \quad \Rightarrow \quad \text{the answer is B.}$$

2017.4. Notice that at any time during the motion, the speed of a given car is equal to the car directly in front (to the right) of it 0.2 s ago. As a corollary, the gap between any two adjacent cars is equal to the distance spanned by the next gap 0.2 s earlier. A quick observation shows that the gap between any two cars begins at a constant x_i, increases over a short time, then remains constant again at some distance $x_f > x_i$. Armed with this description, we examine each figure individually.

1. Figure I shows the beginning of this process, when each gap is at its minimum distance x_i.

2. Figure II shows an intermediate stage, as the second and third gaps are increasing while the first gap is already at its maximum distance.

3. Figure III does not show a step of this process, as the gaps to the right (in front) are less than the gaps behind them, contradicting the fact that the gaps increase over time.

4. Figure IV shows the end of this process, when all gaps are at their final distance x_f.

Thus, the answer choice that most accurately describes this pattern is B.

2017.5. Looking at the answer choices, we can immediately eliminate D as a viable option, as when h is small, the projectile will have a much greater horizontal displacement with positive θ. If we quickly examine the other answers, we can see that this problem asks us to determine how the optimal angle θ compares to $45°$.

We can divide the motion of the projectile up into a first phase, when the projectile has elevation greater than h, and a second phase, when the elevation is less than h. Let the total distance traveled be $d(\theta) = d_1(\theta) + d_2(\theta)$, where d_1 and d_2 are the distances traveled during the first and second phases, respectively. The first phase is a frictionless, level setup, so the projectile travels total distance[1]

$$d_1(\theta) = \frac{v_0^2 \sin 2\theta}{g}.$$

In particular, note that $d_1(\theta)$ achieves its maximum when $\theta = 45°$, and it is decreasing on the interval $(45°, 90°]$. Furthermore, at the beginning of the second phase, the velocity of the projectile equals v_0 by conservation of energy, but its vertical component is flipped in direction. A larger angle θ will lead

[1]This is sometimes known as the *range equation* and can be derived from basic principles of projectile motion.

to a greater angle of attack on the descent, reducing both flight time and horizontal speed. Thus, the distance $d_2(\theta)$ is strictly decreasing in θ.

Armed with these observations, the total distance function $d(\theta)$ is strictly decreasing on the interval $[45°, 90°]$, so it cannot achieve its maximum when $\theta \geq 45°$. Thus, the answer is C.

2017.6. Let the x-coordinate of the top supporting string be 0. We will calculate torques about this axis for both crossbeams. Let the tensions of the connecting strings at positions -0.1 m and 0.2 m be T_1 and T_2, respectively. Letting rotation in the counterclockwise direction be positive, we can set the torques on the two beams to be zero to get

$$(0.4\,\mathrm{m})M_1g + (0.1\,\mathrm{m})T_1 - (0.2\,\mathrm{m})T_2 - (0.5\,\mathrm{m})M_2g = 0,$$

$$(0.2\,\mathrm{m})M_3g - (0.1\,\mathrm{m})T_1 + (0.2\,\mathrm{m})T_2 - (0.4\,\mathrm{m})M_4g = 0.$$

If we add these two equations together, the tensions cancel out, so we can then divide out $(0.1\text{ m})g$ on both sides to get[2]

$$4M_1 - 5M_2 + 2M_3 - 4M_4 = 0.$$

Plugging in the known masses gives $M_3 = 700$ g, so the answer is E.

2017.7. The kinetic energy of the train at time t is given by $\frac{1}{2}(M + \rho t)v^2$. This is linear in t, and the slope is $\frac{1}{2}\rho v^2$. Thus, the answer is D.

2017.8. Over a short time interval Δt, the momentum p of the train increases by

$$\Delta p = \Delta m v = \rho v \Delta t.$$

The impulse generated by the train's engine over this time period is equal to its change in momentum, according to Newton's Second Law. This means that the force produced by the train's engine is

$$F = \frac{J}{\Delta t} = \frac{\Delta p}{\Delta t} = \rho v.$$

Then, the power generated by the engine is

$$P = Fv = \rho v^2 \quad \Rightarrow \quad \text{the answer is E.}$$

Note: The answer to this problem is greater than the answer to Problem 2017.7, as some of the engine's power is lost due to the constant inelastic collisions between the snow and the train.

[2] Alternatively, we could have just considered the external torque on the system as a whole, which would lead to the same equation.

2017.9. This problem is very tricky. Gauge pressure at any point in water is dependent solely on its depth h. It is calculated using the formula $P = \rho g h$, where ρ is the density of water.

Knowing this, it is easy to assume at first glance that the force $F = PA$ is equal for all flasks and move on. However, this is incorrect, as the problem specifies that an equal *volume* of water is poured into each flask. Since the flasks have differing cross-sectional areas, the height of the water is greatest in flask C and least in flask B. The pressures are proportional to depth and behave in the same way, so the answer is D.

2017.10. Gauge pressure is defined as pressure referenced against the ambient atmospheric pressure. The larger part of the milk container is open to the atmosphere, so the gauge pressure is zero at height H in the water. Thus, at the bottom of the jug, $P = \rho g H$, so the answer is B.

Note: This means that the gauge pressure at the surface of the water in the handle is negative. Here, there is a region of low-pressure air trapped by the plug that acts as a suction, pulling water up into the handle.

2017.11. In the absence of external forces, the rod rotates about its center of mass. For the rod to not rotate after the collision, the force due to the impact should be applied at the center of mass. This is equal to the weighted average of the two sections' individual centers of mass (rod and sphere), so

$$h = \frac{m_1 h_1 + m_2 h_2}{m_1 + m_2} = \frac{m \cdot 0 + 2m \cdot L/2}{m + 2m} = \frac{L}{3}.$$

Thus, the answer is B.

2017.12. No matter the magnitude of the applied force, as long as it is at the center of mass, the net torque will be zero. Therefore, the answer is E.

2017.13. We present two approaches to this problem.

Approach 1: In the limit, as $m \gg M$, the lighter mass of M is negligible, so the acceleration of the heavier mass is approximately $-g$. Then, the acceleration of the lighter mass is close to $+g$, so the net force on it is

$$F_{net} = T - Mg = Mg.$$

Solving, we see that in the limit $m \to \infty$, the tension approaches $2Mg$. Thus, T increases, approaching a finite constant, so the answer is D.

Approach 2: We can also determine a formula for the tension as a function of m. Treating the two masses together as a system, their acceleration is

$$a = \frac{F}{m} = \frac{(M+m)g - Mg}{2M+m} = \frac{m}{2M+m}g.$$

This must also be the acceleration of each block individually. Considering the forces acting on the block with mass M, we get that

$$T - Mg = Ma \quad \Rightarrow \quad T = \frac{2M(M+m)}{2M+m}g = 2Mg\left(1 - \frac{M}{2M+m}\right).$$

As m increases, the fraction $\frac{M}{2M+m}$ decreases, approaching a horizontal asymptote at 0. Thus, the tension increases to a finite constant $2Mg$, so the answer is D.

2017.14. We present two approaches to this problem.

Approach 1: Let the object's moment of inertia about its center of mass be $I_c = cMR^2$ for constant c. Notice that c is proportional to the ratio of rotational to translational kinetic energy of the ball. Then, objects with lower values of c will have greater proportions of kinetic energy going into translational movement, leading to higher acceleration. The solid sphere has the lowest value of this parameter, $c = 2/5$, so the answer is A.

Approach 2: Consider an object with circular cross-section and moment of inertia $I_c = cMR^2$ relative to its center of mass axis. By the parallel axis theorem, it has moment of inertia $I = (1+c)MR^2$ about the axis of rotation passing through its contact point with the inclined plane.

An object that rolls without slipping can have its instantaneous motion described as a rotation about the contact-point axis, with zero translational velocity. Then, the acceleration of the center of mass is given by

$$a = \alpha R = \frac{\tau R}{I} = \frac{MgR^2 \sin\theta}{(1+c)MR^2} = \frac{g\sin\theta}{1+c}.$$

Thus, the object with the lowest coefficient c will have the largest acceleration down the incline. In this case, this is the solid sphere, where $c = 2/5$, so the answer is A.

Note: The mass M and radius R both cancel out and are not in the final formula for a. This means, in particular, that the answer to this problem would be the same even if the objects had different masses, densities, and radii.

2017.15. By the kinematic formulas, we have that

$$v = \sqrt{v_0^2 + 2ad} = \sqrt{\frac{2ah}{\sin\theta}} \propto \sqrt{a}.$$

Thus, we are looking for the object with the highest value of a. Given the result from the previous problem, this only depends on the coefficient c, but since all answer choices are solid spheres, $c = 2/5$ for each of them. This is independent of the spheres' masses, radii, and densities. Thus, they all have the same speed, so the answer is E.

2017.16. We present two approaches to this problem.

Approach 1: This approach uses only basic trigonometry.

Let the velocity of the rod's end in contact with the wall be $\vec{u}$. For the length of the rod to stay constant, the component of $\vec{v}$ parallel to the rod must equal the component of $\vec{u}$ parallel to the rod. This means that

$$v\cos\theta = u\cos(\alpha-\theta) \quad \Rightarrow \quad u = v\frac{\cos\theta}{\cos(\alpha-\theta)}.$$

Thus, the answer is D.

Approach 2: This approach requires some basic calculus. It is presented here to show a variety of methods, but you can feel free to skip it.

Let the length of the rod be ℓ. Also, let the distance between the corner and the rod's contact with the floor be x_1, and the distance between the corner and the rod's contact with the wall be x_2. By the Law of Sines, we have

$$\frac{x_1}{\sin(\alpha-\theta)} = \frac{x_2}{\sin\theta} = \frac{\ell}{\sin\alpha}.$$

As the rod falls, ℓ and α do not change, so we can replace the last expression above, $\ell/\sin\alpha$, with a constant c. Then, our equations become

$$\begin{aligned} x_1 &= c\sin(\alpha-\theta), \\ x_2 &= c\sin\theta. \end{aligned}$$

The derivative of $\sin\theta$ with respect to θ is just $\cos\theta$, so by the chain rule,

$$\frac{dx_2/dt}{dx_1/dt} = \frac{dx_2/d\theta}{dx_1/d\theta} = \frac{c\cos\theta}{-c\cos(\alpha-\theta)} = -\frac{\cos\theta}{\cos(\alpha-\theta)}.$$

However, since $dx_1/dt = v$, the speed of the end in contact with the wall is

$$|dx_2/dt| = v\frac{\cos\theta}{\cos(\alpha-\theta)}.$$

Thus, as before, the answer is D.

2017.17. The average velocity during the last 1.0 seconds is 60 m/s. Because the acceleration is constant, this average velocity also equals the instantaneous velocity 0.5 s before impact, so the velocity at impact is $60\,\text{m/s} + (0.5\,\text{s})g = 65\,\text{m/s}$. However, by the kinematic formulas, we know that

$$v_f^2 - v_i^2 = 2ad \quad \Rightarrow \quad v_i = \sqrt{v_f^2 - 2gh}.$$

Plugging in the known values and calculating yields

$$v_i = \sqrt{(65\text{ m/s})^2 - 2(10\text{ m/s}^2)(180\text{ m})} = 25\text{ m/s}.$$

Thus, the answer is B.

2017.18. The total acceleration of any point on the disk is given by the sum of three parts: the centripetal acceleration $a_c = r\omega^2$, the tangential acceleration $a_t = r\alpha$, and the translational acceleration a of the disk. As vectors, $\vec{a}_c$ points toward the center of the disk, $\vec{a}_t$ points in the direction of rotation, and $\vec{a}$ always points right, in the direction of F. For the total acceleration to be zero, $\vec{a}_c + \vec{a}_t$ must point due left to cancel out with $\vec{a}$. We then consider each region of the disk individually:

1. In Region I, $\vec{a}_c$ points to the bottom left while $\vec{a}_t$ points to the bottom right, so their sum points roughly down.
2. In Region II, $\vec{a}_c$ points to the bottom right while $\vec{a}_t$ points to top right, so their sum points roughly right.
3. In Region III, $\vec{a}_c$ points to the top right while $\vec{a}_t$ points to the top left, so their sum points roughly up.
4. In Region IV, $\vec{a}_c$ points to the top left while $\vec{a}_t$ points to the bottom left, so their sum points roughly left.

This qualitative analysis rules out answer choices A, B, and C. The next question is: does there exist a point in IV with the proper values of $\vec{a}_c$, $\vec{a}_t$, and $\vec{a}$ such that the total acceleration is zero?

It turns out that the answer to this question is yes! Let's work in polar coordinates (r, θ). The magnitude of $\vec{a}_c + \vec{a}_t$ depends on r, while its direction depends on θ. In particular, if we can find a value of r such that the magnitude of this sum equals the magnitude of $\vec{a}$, then we can adjust θ appropriately so that the vector points left and cancels out with it.

Since the centripetal and tangential accelerations are perpendicular, the magnitude of their sum is

$$\|\vec{a}_c + \vec{a}_t\| = \sqrt{\|\vec{a}_c\|^2 + \|\vec{a}_t\|^2} = r\sqrt{\omega^4 + \alpha^2}.$$

Meanwhile, since the disk rolls without slipping, the magnitude of $\vec{a}$ is αR. By the intermediate value theorem, there exists some $r \in [0, R]$ such that the magnitude $\|\vec{a}_c + \vec{a}_t\|$ is equal to αR.

After finding this value of r, we can adjust θ appropriately, as discussed before, to rotate $\vec{a}_c + \vec{a}_t$ so that its direction is due left. At this point, the vectors $\vec{a}_c + \vec{a}_t$ and $\vec{a}$ are equal and opposite, so their sum is zero. Thus, we've shown that a point with zero total acceleration exists and lies in Region IV, so the answer is D.

2017.19. Looking at the slopes in the graph, you can see that as the puck moves up the ramp, its acceleration is $a_1 = 2a$, and when it goes down the ramp, its acceleration is $a_2 = 2a/3$, for some value of a that depends on the graph's scale. In other words, there is a 3 : 1 ratio between the accelerations. This ratio can be explained by the change in the direction of kinetic friction:

$$\begin{aligned} a_1 &= g\sin\theta + \mu g\cos\theta, \\ a_2 &= g\sin\theta - \mu g\cos\theta. \end{aligned}$$

Dividing these equations, we compute the ratio between the accelerations to be equal to

$$\frac{3}{1} = \frac{a_1}{a_2} = \frac{g\sin\theta + \mu g\cos\theta}{g\sin\theta - \mu g\cos\theta} = \frac{1+\mu\sqrt{3}}{1-\mu\sqrt{3}}.$$

It follows that $\mu\sqrt{3} = 1/2$, so $\mu = \frac{1}{2\sqrt{3}} \approx 0.29$. Thus, the answer is D.

2017.20. By conservation of momentum, the speed of the objects after collision is

$$v_1 = \frac{mv_0}{m+M}.$$

Then, the fractional momentum transfer is given by

$$f = \frac{Mv_1}{mv_0} = \frac{M}{m+M}.$$

This fraction approaches its maximum when $m \ll M$, so the answer is A.

2017.21. The fractional momentum transfer is maximized when the momentum of the particle with mass M is maximal, or equivalently, when the final momentum of the mass-m particle is minimal. One good candidate for this is when $m \ll M$. In this scenario, the mass-M particle acts effectively as a stationary wall, so the mass-m particle should bounce off at velocity $-v_0$ and momentum $-mv_0$. In fact, this is the optimal scenario, as by conservation of energy, the final speed of the mass-m particle can be no greater than v_0.

From this, we know that the fractional momentum transfer when $m \ll M$ is maximal and is given by

$$f_{max} = \frac{p_0 - p_1}{p_0} = \frac{mv_0 - (-mv_0)}{mv_0} = 2.$$

Thus, the answer is D.

2017.22. The fractional energy transfer can be no greater than 1 without violating conservation of energy. In the special case where two objects with equal masses collide elastically along a line, their velocities are simply swapped[3]. Thus, when $m = M$, the fractional energy transfer is 1, so the answer is C.

2017.23. The speed of a transverse wave on a spring is given by $v = \sqrt{T/\mu}$, where T is the magnitude of tension and μ is the mass density of the spring. When the spring is stretched from 10 m to 20 m, its tension is approximately doubled (by Hooke's Law) while its mass density is halved, so the wave speed is doubled overall. However, the length of the spring is also doubled, so the travel time remains approximately the same. Thus, the answer is C.

2017.24. We present two approaches to this problem.

Approach 1: We can use the answer choices to our advantage. First, consider the limiting case when $M \gg m$. Then, the box acts as a stationary wall, so the maximum compression of the spring can be calculated by equating kinetic and potential energy to get

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 \quad \Rightarrow \quad x = v\sqrt{\frac{m}{k}}.$$

This rules out answer choices C, D, and E, since they do not behave this way as $M \to \infty$. Among the last two options, notice that B does not include M in its expression, which is highly unlikely, so we can rule it out. Thus, the answer is A.

Approach 2: At the time of maximum compression, the ball has zero velocity relative to the box (because if they were still moving with respect to each other, the distance would not be at its minimum), and momentum is conserved. This is precisely the same scenario as a perfectly inelastic collision! The only difference is that the energy is stored within the spring, instead of lost to heat and sound. The kinetic energy of the system at this point is

$$\frac{1}{2}(m+M)\left(\frac{mv}{m+M}\right)^2 = \frac{m}{m+M} \cdot \frac{1}{2}mv^2.$$

[3] You can verify this easily by checking that both total momentum and kinetic energy are conserved by swapping the velocities.

However, the initial kinetic energy was $\frac{1}{2}mv^2$. The change in kinetic energy is stored as potential energy in the spring, which can be calculated as the difference

$$U = \frac{1}{2}mv^2 - \frac{m}{m+M} \cdot \frac{1}{2}mv^2 = \frac{M}{m+M} \cdot \frac{1}{2}mv^2.$$

We finish with the elastic potential energy formula, which yields

$$U = \frac{1}{2}kx^2 \quad \Rightarrow \quad x = \sqrt{\frac{2U}{k}} = v\sqrt{\frac{mM}{(m+M)k}}.$$

Thus, the answer is A.

2017.25. We present two approaches to this problem.

Approach 1: The *vis-viva* equation[4] states that in any Keplerian orbit, the speed of the orbiting body is given by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

where r is the distance from the central body, a is the length of the semi-major axis, M is the central body's mass, and G is the universal gravitational constant. Since the sum of the distances from a point on the ellipse to the two foci is constant, we can solve for the length SP, as

$$SP + \sqrt{SP^2 + a^2} = 2a \quad \Rightarrow \quad SP = \frac{3}{4}a.$$

Then, by the vis-viva equation,

$$\frac{v_2^2}{v_1^2} = \frac{\frac{2}{a/2} - \frac{1}{a}}{\frac{2}{3a/4} - \frac{1}{a}} = \frac{4-1}{8/3-1} = \frac{9}{5}.$$

Thus, $v_2 = \frac{3}{\sqrt{5}}v_1$, so the answer is A.

Approach 2: Like the previous solution, we begin by using the Pythagorean theorem to calculate $SP = 3a/4$. The key to this approach is that if we can express v_2 in terms of G, M, and a, then we can use conservation of energy to calculate v_1.

We will write a system of equations to solve for v_1. Let v_3 be the speed of the planet at the apogee. By conservation of energy, we get the equation

$$\frac{1}{2}mv_2^2 - \frac{GMm}{a/2} = \frac{1}{2}mv_3^2 - \frac{GMm}{3a/2}.$$

[4] For more information: `https://en.wikipedia.org/wiki/Vis-viva_equation`.

Also, by conservation of angular momentum (or Kepler's Second Law), we have that

$$\frac{mv_2a}{2} = \frac{3mv_3a}{2} \quad \Rightarrow \quad v_3 = \frac{1}{3}v_2.$$

We can plug this into the energy conservation equation and rearrange to get

$$\frac{4}{9}mv_2^2 = \frac{GMm}{a/2} - \frac{GMm}{3a/2} = \frac{4GMm}{3a},$$

$$v_2^2 = \frac{3GM}{a}.$$

Great, we got a formula for v_2! Now, to calculate v_1, we can use conservation of energy between the perigee and point P to get

$$\frac{1}{2}mv_2^2 - \frac{GMm}{a/2} = \frac{1}{2}mv_1^2 - \frac{GMm}{3a/4}.$$

We can then plug in our expression for v_2^2 and simplify to get

$$v_1^2 = v_2^2 - \frac{4GM}{a} + \frac{8GM}{3a} = GM\left(\frac{3}{a} - \frac{4}{a} + \frac{8}{3a}\right) = \frac{5GM}{3a}.$$

Finally, dividing our formulas for v_2 and v_1 yields

$$\frac{v_2^2}{v_1^2} = \frac{3 \cdot GM/a}{5/3 \cdot GM/a} = \frac{9}{5}.$$

Thus, the answer is A.

Note: In this approach, one may try to skip v_3 entirely and write a similar system of equations for v_1 and v_2. However, this doesn't work, as the velocity of the planet at point P is not perpendicular to SP, so the angular momentum is harder to write in terms of v_1.

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Year 2018

In 2018, two $F = ma$ exams were offered, Exam A and Exam B. ToC

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2018 $F = ma$ Exam A

ToC

AAPT UNITED STATES PHYSICS TEAM
AIP 2018

2018 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

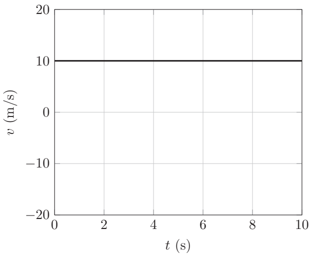
DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2018.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

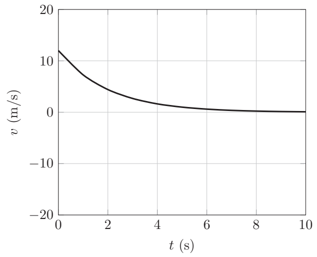
DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

1. Which of the following graphs best shows the velocity versus time of an object originally moving upward in the presence of air friction?

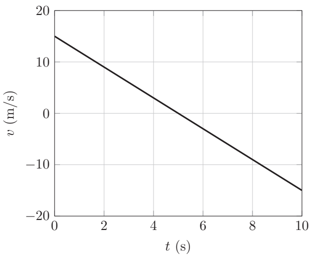
(A)



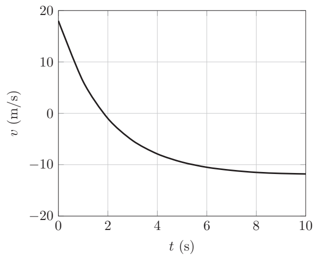
(B)



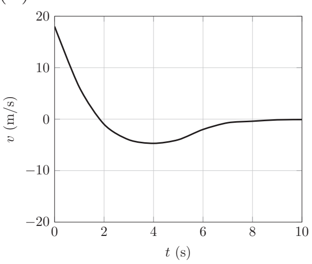
(C)



(D)



(E)



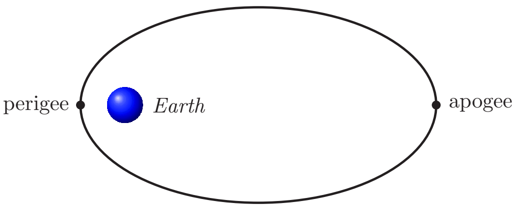
2. A 3.0 kg mass moving at 30 m/s to the right collides elastically with a 2.0 kg mass traveling at 20 m/s to the left. After the collision, the center of mass of the system is moving at a speed of

   (A) 5 m/s
   (B) 10 m/s
   (C) 20 m/s
   (D) 24 m/s
   (E) 26 m/s

3. Ball 1 traveling in the positive x direction strikes an equal mass ball 2 that is originally at rest. All of the following must be true after the collision, except

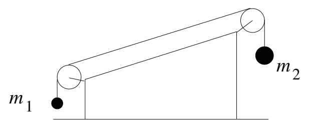
   (A) The total final momentum in the x direction equals the initial momentum of ball 1.
   (B) The total kinetic energy after the collision equals the initial kinetic energy of ball 1.
   (C) The final momentum of the two balls in the y direction adds to zero.
   (D) The final speed of the center of mass of the two balls is equal to half the initial speed of ball 1.
   (E) The balls can't both be at rest after the collision.

4. A satellite is following an elliptical orbit around the Earth. Its engines are capable of providing a one-time impulse of a fixed magnitude. In order to maximize the energy of the satellite, the impulse should be



   (A) directed along the satellite's velocity and applied when the satellite is in its perigee.
   (B) directed along the satellite's velocity and applied when the satellite is in apogee.
   (C) directed toward the Earth and applied when the satellite is in perigee.
   (D) directed toward the Earth and applied when the satellite is in apogee.
   (E) directed away from the Earth and applied when the satellite is in apogee.

5. Two masses are attached with pulleys by a massless rope on an inclined plane as shown. All surfaces are frictionless. If the masses are released from rest, then the inclined plane



   (A) accelerates to the left if $m_1 < m_2$
   (B) accelerates to the right if $m_1 < m_2$
   (C) accelerates to the left regardless of the masses
   (D) accelerates to the right regardless of the masses
   (E) does not move

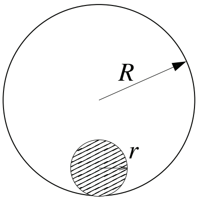
6. A packing crate with mass $m = 115$ kg is slid up a 5.00 m long ramp which makes an angle of 20.0° with respect to the horizontal by an applied force of $F = 1.00 \times 10^3$ N directed parallel to the ramp's incline. A frictional force of magnitude $f = 4.00 \times 10^2$ N resists the motion. If the crate starts from rest, what is its speed at the top of the ramp?

    (A) 4.24 m/s
    (B) 5.11 m/s
    (C) 7.22 m/s
    (D) 8.26 m/s
    (E) 9.33 m/s

7. A car has a maximum acceleration of a_0 and a minimum acceleration of $-a_0$. The shortest possible time for the car to begin at rest, then arrive at rest at a point a distance d away is

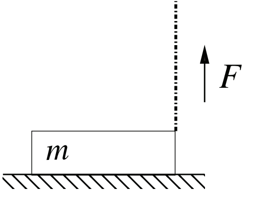
    (A) $\sqrt{d/2a_0}$
    (B) $\sqrt{d/a_0}$
    (C) $\sqrt{2d/a_0}$
    (D) $\sqrt{3d/a_0}$
    (E) $2\sqrt{d/a_0}$

8. A disk of radius r rolls uniformly without slipping around the inside of a fixed hoop of radius R. If the period of the disc's motion around the hoop is T, what is the instantaneous speed of the point on the disk opposite to the point of contact?



    (A) $2\pi(R + r)/T$
    (B) $2\pi(R + 2r)/T$
    (C) $4\pi(R - 2r)/T$
    (D) $4\pi(R - r)/T$
    (E) $4\pi(R + r)/T$

9. A uniform stick of mass m is originally on a horizontal surface. One end is attached to a vertical rope, which pulls up with a constant tension force F so that the center of the mass of the stick moves upward with acceleration $a < g$. The normal force N of the ground on the other end of the stick shortly after the right end of the stick leaves the surface satisfies



(A) $N = mg$

(B) $mg > N > mg/2$

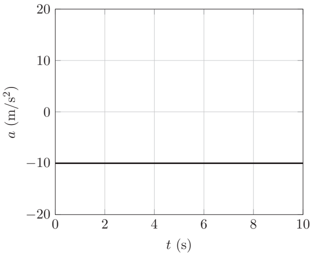
(C) $N = mg/2$

(D) $mg/2 > N > 0$

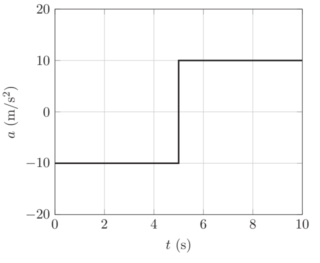
(E) $N = 0$

10. Which of the following graphs best shows the acceleration versus time of an object originally moving upward in the presence of air friction?

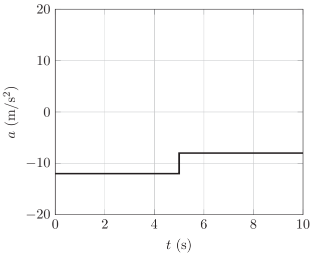
(A)



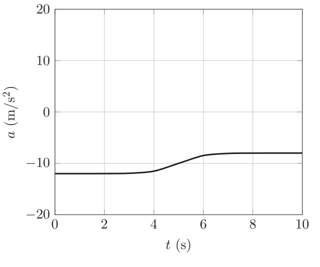
(B)



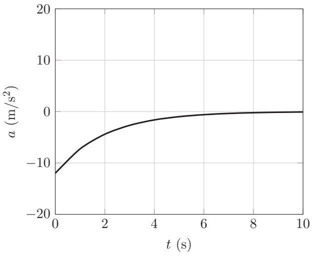
(C)



(D)



(E)



11. A light, uniform, ideal spring is fixed at one end. If a mass is attached to the other end, the system oscillates with angular frequency ω. Now suppose the spring is fixed at the other end, then cut in half. The mass is attached between the two half springs.

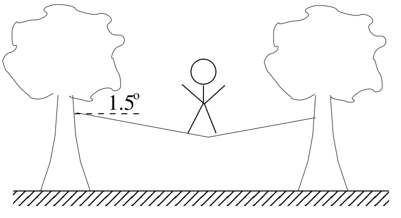
The new angular frequency of oscillations is

(A) $\omega/2$
(B) ω
(C) $\sqrt{2}\omega$
(D) 2ω
(E) 4ω

12. A group of students wish to measure the acceleration of gravity with a simple pendulum. They take one length measurement of the pendulum to be $L = 1.00 \pm 0.05$ m. They then measure the period of a single swing to be $T = 2.00 \pm 0.10$ s. Assume that all uncertainties are Gaussian. The computed acceleration of gravity from this experiment illustrating the range of possible values should be recorded as

(A) 9.87 ± 0.10 m/s^2
(B) 9.87 ± 0.15 m/s^2
(C) 9.9 ± 0.25 m/s^2
(D) 9.9 ± 1.1 m/s^2
(E) 9.9 ± 1.5 m/s^2

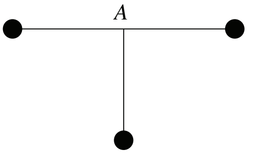
13. A massless cable of diameter 2.54 cm (1 inch) is tied horizontally between two trees 18.0 m apart. A tightrope walker stands at the center of the cable, giving it a tension of 7300 N. The cable stretches and makes an angle of 1.50° with the horizontal.



The Young's modulus is defined as the ratio of stress to strain, where stress is the force applied per unit area and strain is the fractional change in length $\Delta L/L$. The cable's Young's modulus is

(A) 1.5×10^6 N/m^2
(B) 2.0×10^8 N/m^2
(C) 2.2×10^9 N/m^2
(D) 2.4×10^{10} N/m^2
(E) 4.2×10^{10} N/m^2

14. Three identical masses are connected with identical rigid rods and pivoted at point A. If the lowest mass receives a small horizontal push to the left, it oscillates with period T_1. If it instead receives a small push into the page, it oscillates with period T_2. The ratio T_1/T_2 is

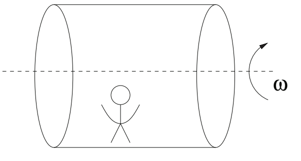


(A) $1/2$
(B) 1
(C) $\sqrt{3}$
(D) $2\sqrt{2}$
(E) $2\sqrt{5}$

15. A satellite is in a circular orbit about the Earth. Over a long period of time, the effects of air resistance decrease the satellite's total energy by 1 J. The kinetic energy of the satellite

    (A) increases by 1 J.
    (B) remains unchanged.
    (C) decreases by $\frac{1}{2}$ J.
    (D) decreases by 1 J.
    (E) decreases by 2 J.

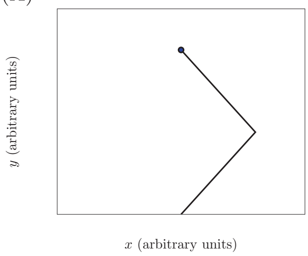
16. A cylindrical space station produces 'artificial gravity' by rotating with angular frequency ω. Consider working in the reference frame rotating with the space station. In this frame, an astronaut is initially at rest standing on the floor, facing in the direction that the space station is rotating. The astronaut jumps up vertically relative to the floor of the space station, with an initial speed less than that of the speed of the floor. Just after leaving the floor, the motion of the astronaut, relative to the space station floor,



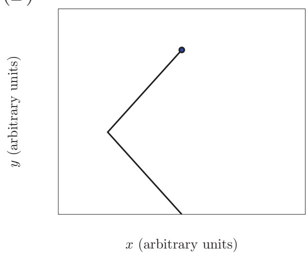
    (A) always has a component of acceleration directed toward the floor, and they land at the same point they jumped from.
    (B) always has a component of acceleration directed toward the floor, and they land in front of the point they jumped from.
    (C) always has a component of acceleration directed toward the floor, and they land behind the point they jumped from.
    (D) has a component of acceleration directed away from the floor, and they land behind the point they jumped from.
    (E) has a zero acceleration relative to the floor, and the astronaut never reaches the floor again.

17. A stream of sand is dropped out of a helicopter initially moving at a constant speed v to the right. The helicopter suddenly turns and begins moving a constant speed v to the left. Neglecting air resistance on the sand, what is the shape of the stream of sand, as viewed from the ground? The black dot represents the helicopter.

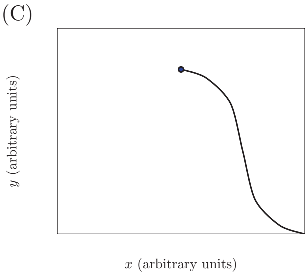
(A)



(B)



(C)



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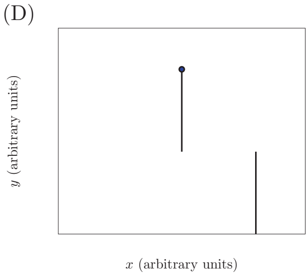
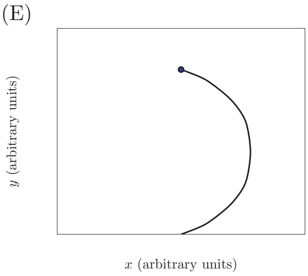


(E)



18. A mass m is attached to a thin rod of length ℓ so that it can freely spin in a vertical circle with period T. The difference in the tensions in the rod when the mass is at the top and the bottom of the circle is

(A) $6mg^2T^2/\ell$
(B) $4\pi mg^2T^2/\ell$
(C) $6mg$
(D) $\pi^2 m\ell/T^2$
(E) $4\pi m\ell/T^2$

19. Raindrops with a number density of n drops per cubic meter and radius r_0 hit the ground with a speed v_0. The resulting pressure on the ground from the rain is P_0. If the number density is doubled, the drop radius is halved, and the speed is halved, the new pressure will be

    (A) P_0
    (B) $P_0/2$
    (C) $P_0/4$
    (D) $P_0/8$
    (E) $P_0/16$

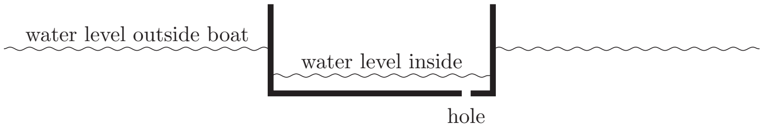
20. A spring stretched to double its unstretched length has a potential energy U_0. If the spring is cut in half, and each half spring is stretched to double its unstretched length, then the total potential energy stored in the two half springs will be

    (A) $4U_0$
    (B) $2U_0$
    (C) U_0
    (D) $U_0/2$
    (E) $U_0/4$

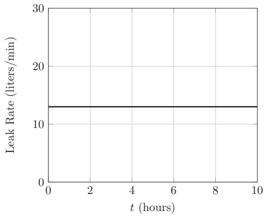
21. A ping-pong ball (a hollow spherical shell) with mass m is placed on the ground with initial velocity v_0 and zero angular velocity at time $t = 0$. The coefficient of friction between the ping-pong ball and the ground is $\mu_s = \mu_k = \mu$. The time the ping-pong ball begins to roll without slipping is

    (A) $t = (2/5)v_0/\mu g$
    (B) $t = (2/3)v_0/\mu g$
    (C) $t = v_0/\mu g$
    (D) $t = (5/3)v_0/\mu g$
    (E) $t = (3/2)v_0/\mu g$

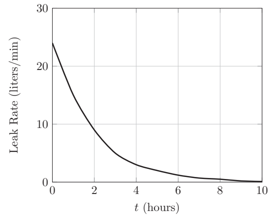
22. A small hole is punched into the bottom of a rectangular boat, allowing water to enter the boat. As the boat sinks into the water, which of the following graphs best shows how the rate water flows through the hole varies with time? Assume that the boat remains horizontal as it sinks.



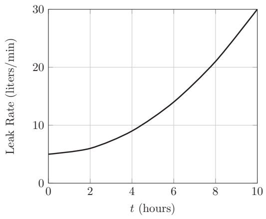
(A)



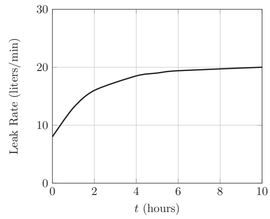
(B)



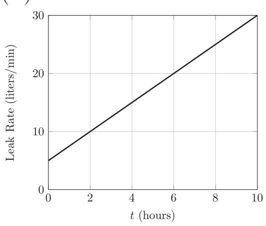
(C)



(D)

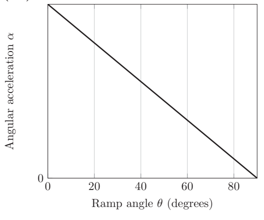


(E)

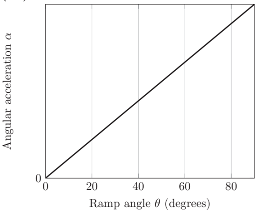


23. The coefficients of static and kinetic friction between a ball and an ramp are $\mu_s = \mu_k = \mu$. The ball is released from rest at the top of the ramp. Which of the following graphs best shows the rotational acceleration of the ball about its center of mass as a function of the angle of the ramp?

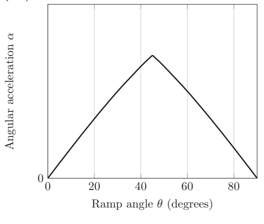
(A)



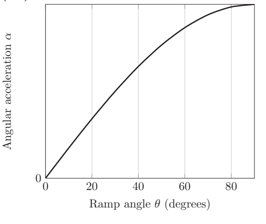
(B)



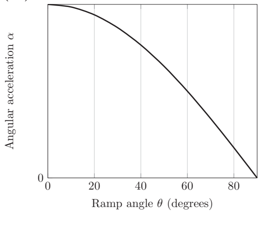
(C)



(D)



(E)



24. A mass is attached to one end of a rigid rod, while the other end of the rod is attached to a fixed horizontal axle. Initially the mass hangs at the end of the rod and the rod is vertical. The mass is given an initial kinetic energy K. If K is very small, the mass behaves like a pendulum, performing small-angle oscillations with period T_0. As K is increased, the period of the motion for the mass

(A) remains the same.
(B) increases, approaching a finite constant.
(C) decreases, approaching a finite non-zero constant.
(D) decreases, approaching zero.
(E) initially increases, then decreases.

25. Alice and Bob are working on a lab report. Alice measures the period of a pendulum to be 1.013 ± 0.008 s, while Bob independently measures the period to be 0.997 ± 0.016 s. Alice and Bob can combine their measurements in several ways.

    1: Keep Alice's result and ignore Bob's

    2: Average Alice's and Bob's results

    3: Perform a weighted average of Alice's and Bob's results, with Alice's result weighted 4 times more than Bob's

    How are the uncertainties of these results related?

    (A) Method 1 has the lowest uncertainty, and method 2 has the highest

    (B) Method 3 has the lowest uncertainty, and method 2 has the highest

    (C) Method 2 has the lowest uncertainty, and method 1 has the highest

    (D) Method 3 has the lowest uncertainty, and method 1 has the highest

    (E) Method 1 has the lowest uncertainty, and method 3 has the highest

Answers, Problem Difficulty, and Topics

ToC

2018-A.1	D	⋆	air friction, projectile motion
2018-A.2	B	⋆	conservation of linear momentum, collisions
2018-A.3	B	⋆	collisions
2018-A.4	A	⋆ ⋆	Kepler's Laws, kinetic energy
2018-A.5	E	⋆	conservation of linear momentum
2018-A.6	A	⋆ ⋆	kinetic friction, work, energy
2018-A.7	E	⋆	linear motion
2018-A.8	D	⋆ ⋆	rolling motion
2018-A.9	B	⋆ ⋆	circular motion, torque
2018-A.10	E	⋆	air friction, projectile motion
2018-A.11	D	⋆ ⋆	springs, oscillations
2018-A.12	D	⋆ ⋆ ⋆	measurement error analysis
2018-A.13	E	⋆	forces
2018-A.14	C	⋆ ⋆	physical pendulum, oscillations
2018-A.15	A	⋆ ⋆	conservation of energy, gravitation
2018-A.16	B	⋆ ⋆ ⋆	rotating reference frame, Coriolis force
2018-A.17	D	⋆ ⋆	inertia, projectile motion
2018-A.18	C	⋆ ⋆ ⋆	circular motion, centripetal force, energy
2018-A.19	E	⋆ ⋆	collisions, pressure
2018-A.20	C	⋆	springs, Hooke's Law
2018-A.21	A	⋆ ⋆	torque, kinetic friction
2018-A.22	A	⋆ ⋆ ⋆	Archimedes' Principle
2018-A.23	C	⋆ ⋆	rolling motion, friction
2018-A.24	E	⋆ ⋆	simple pendulum
2018-A.25	B	⋆ ⋆	measurement error analysis

Solutions for Exam A

ToC

2018.1. Air friction acts on the object and exerts a changing force opposing its motion. When the object moves upward, air resistance causes it to accelerate downward faster than g, and when it moves downward, air resistance counteracts the force of gravity and leads to the object to reach its terminal velocity. Graph D shows this behavior, as the object initially accelerates downward rapidly when its velocity is positive, then slows down as it changes direction. Eventually, the graph reaches a horizontal nonzero asymptote where gravity and friction are in equilibrium. The other four graphs are incorrect, as

- A shows the velocity of the object as constant, but it should be decreasing.
- B has a horizontal asymptote at zero, but the ball cannot be in equilibrium when it is stationary due to the force of gravity.
- C shows the graph if air friction were negligible and acceleration was constant.
- E shows a period of positive acceleration after 4 s, but the velocity of the object should be monotonically decreasing. Also, the horizontal asymptote is at the wrong location.

Therefore the correct answer is D.

2018.2. Elastic collisions, or indeed, any interactions isolated from external forces, preserve the velocity of the center of mass by Newton's First Law. The velocity of the center of mass after collision is equal to the velocity $\vec{v}$ of the center of mass before collision. This is simply the weighted average of the initial velocities of the objects[1], so

$$v = \left\| \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \right\| = \left| \frac{(3.0\text{ kg})(30\text{ m/s}) + (2.0\text{ kg})(-20\text{ m/s})}{3.0\text{ kg} + 2.0\text{ kg}} \right| = 10\text{ m/s}.$$

Thus, the answer is B.

2018.3. Except for B, the statements of all the other answer choices follow from the conservation of momentum in a closed system. However, kinetic energy is not necessarily conserved in this case, as the collision in the problem statement could be inelastic. Therefore, the answer is B.

[1]This comes from momentum, as $m_c v_c = m_1 v_1 + m_2 v_2$.

2018.4. First, note that the impulse should be applied in the same direction as the ship's velocity to maximize the work done on the ship. By Newton's Second Law, impulse equals change in momentum, so the ship's change in velocity from the engine thrust is fixed; call this Δv. Also, let the instantaneous velocity of the ship before the engine thrust be v_0, and its mass be m. The change in kinetic energy is

$$\begin{aligned}\Delta K &= \frac{1}{2}m((v_0+\Delta v)^2 - v_0^2)\\ &= \frac{1}{2}m(\Delta v^2 + 2v_0\Delta v) \;=\; C + (m\Delta v)v_0,\end{aligned}$$

for some constant C. This is highest when v_0 is at its maximum, or at the closest point by Kepler's Second Law, so the answer is A.

2018.5. Consider the system of the two masses and the inclined plane. Initially, the only external forces on this system, gravity and the normal force, act in strictly the y-direction. Then the x-component of the momentum is conserved, and the inclined plane does not move. The answer is E.

2018.6. In this problem, the net force applied to the packing crate is constant, so it is natural to take an energy approach. The net force in the direction up the incline, before taking into account gravity, is

$$F - f = 1000\ \text{N} - 400\ \text{N} = 600\ \text{N}.$$

Then the total work done by nonconservative forces on the crate is

$$W = F\ell = (600\ \text{N})(5.00\ \text{m}) = 3000\ \text{J}.$$

Some of this work goes into increasing the gravitational potential energy of the crate by

$$\Delta U = mg\Delta h = mg\ell\sin\theta = (115\ \text{kg})(10\ \text{m/s}^2)(5.00\ \text{m})\sin 20.0^\circ = 1967\ \text{J}.$$

The rest of this work goes into the increasing the final kinetic energy of the crate to be $K = W - \Delta U = 1033$ J, so

$$K = \frac{1}{2}mv_f^2 \quad\Rightarrow\quad v_f = \sqrt{\frac{2K}{m}} = 4.24\ \text{m/s}.$$

Thus, the answer is A.

Note: This solution is general in that it remains valid even when the shape of the ramp is not a straight line, as long as the frictional force remains of constant magnitude, and the external force F stays directed tangent to the curve. In particular, we could be looking at a roller coaster, going up and down, as long as the above conditions are met.

2018.7. Intuitively, for this problem, we want the car to travel as fast as possible on average, but we cannot go so fast that our limited braking capacity causes us to overshoot our target. From this, we can see that the best strategy is to maintain maximum acceleration for the first half of the time, then switch to maximum deceleration for the second half.

Since the car's velocity in the first and second halves are symmetric with respect to time, it travels a distance of $d/2$ in the first half of the time. Then, letting t be the total time, we have, by the constant acceleration kinematic formula,

$$\frac{d}{2} = \frac{1}{2}a_0(t/2)^2 \quad \Rightarrow \quad t = 2\sqrt{d/a_0}.$$

Thus, the answer is E.

2018.8. This problem is quite tricky in that it involves the common configuration of a disk rolling tangent to a circle. Indeed, a similar problem appeared as #14 on the 2015 AMC 10A, a high school mathematics competition.

The issue is that tracing the path of any fixed point on the small disk produces an interesting shape that's difficult to analyze[2], and additionally, the speed of the point is not constant. This difficulty can be averted by instead considering the path of the center of the small disk, which is simply a circle of radius $R-r$ concentric with the large circle. Since the disk rolls uniformly without slipping, the speed of its center is

$$v_c = \frac{2\pi(R-r)}{T}.$$

Furthermore, as the disk rolls without slipping, its instantaneous movement can be described as rotation about an axis passing through the point of contact with the hoop. Since the point opposite the point of contact is exactly twice the distance from the axis of rotation as the center, its instantaneous velocity must be double that of the center, so

$$v = 2v_c = 4\pi(R-r)/T \quad \Rightarrow \quad \text{the answer is D.}$$

Note 1: Alternatively, as the disk rolls without slipping, its angular speed is $\omega = v_c/r$, so the speed of the point opposite to the point of contact is $v = v_c + r\omega = 2v_c$, and we can proceed as before.

Note 2: As a student in our class suggested, if this were a later question and you were running out of time, it would be a good idea to intuitively guess the answer choice D, since it is the only one with $R-r$ to refer to the center of mass.

[2] This particular shape is known as a hypotrochoid.

2018.9. To solve this problem, we need to solve equations for both angular and translational acceleration. Let ℓ be the length of the stick. Since the stick initially rotates about an axis passing through its left end, we have that

$$a = r\alpha = \alpha\ell/2 \quad \text{and} \quad I = \frac{1}{3}m\ell^2.$$

The total torque on the stick about this axis is then

$$\tau = F\ell - \frac{1}{2}mg\ell.$$

Using this, we can calculate the angular acceleration to be

$$\alpha = \frac{\tau}{I} = \frac{1}{\ell}\left(3 \cdot \frac{F}{m} - \frac{3}{2}g\right).$$

Combining this with the previous expression relating a to α, we see that

$$2a = \alpha\ell = 3 \cdot \frac{F}{m} - \frac{3}{2}g.$$

However, the net force is $F - mg + N$, so using Newton's Second Law,

$$a = \frac{F_{net}}{m} = \frac{F}{m} - g + \frac{N}{m} \quad \Rightarrow \quad 3a = 3 \cdot \frac{F}{m} + 3 \cdot \frac{N}{m} - 3g.$$

Subtracting the two above equations to cancel out F yields

$$a = 3 \cdot \frac{N}{m} - 3g + \frac{3}{2}g \quad \Rightarrow \quad N = \frac{1}{2}mg + \frac{1}{3}ma.$$

Since $a < g$ as given in the problem statement, the answer is B.

2018.10. Another air friction question! We follow the same line of reasoning as in Problem 1 in this exam, but this time, we are looking for the graph of acceleration over time. Key features to note are that

- The graph should reach a horizontal asymptote at zero as the object approaches its terminal velocity.
- There should be no sharp bumps in the graph, as the velocity-over-time graph is smooth.
- The acceleration should always be negative, since the net force (gravity and air resistance) is downward.

Only graph E has all of these features (or even just the first feature), so the answer is E.

2018.11. This problem tests a key fact often seen on $F = ma$ tests, which is that the spring constant is inversely proportional to a spring's length. Then, consider the elastic restoring force on a mass of distance Δx from the equilibrium in both scenarios. Assume the force in the first scenario is $F = -k\Delta x$. Then, in the second scenario it is four times as large, i.e. $F = -4k\Delta x$, as there are now two springs working together, each with double the strength of the original. Since the effective spring constant k is multiplied by four, the angular frequency $\omega = \sqrt{k/m}$ is doubled, so the answer is D.

2018.12. This problem asks us to estimate the uncertainty (given by standard deviation) Δg of the value of g from given measurements and associated uncertainties. The formula for the period of a simple pendulum is

$$T = 2\pi\sqrt{L/g} \quad \Rightarrow \quad g = 4\pi^2 L/T^2.$$

We know uncertainties for measurements of L and T and need to figure out how to combine them. The rules we need here are:

- Sum[3] : If measurements x and y have uncertainties Δx and Δy, the uncertainty of their sum $x + y$ is $\Delta(x + y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.
- Product[4] : If measurements x and y have uncertainties Δx and Δy, the uncertainty of their product xy is $\Delta(xy) = \sqrt{(x\Delta y)^2 + (y\Delta x)^2}$.
- Power[5] : If a measurement of x has uncertainty Δx, then the uncertainty of x^a is $\Delta(x^a) = |a|x^{a-1}\Delta x$.

[3] For rule of sum: Let $x = m_x + e_x$ and $y = m_y + e_y$, where m_x and m_y are the unknown true values of the measured variables x and y, while e_x and e_y are zero-mean Gaussian variables with standard deviations $\Delta e_x = \Delta x \ll m_x$ and $\Delta e_y = \Delta y \ll m_y$. Then, the standard deviation of $x + y$ is

$$\Delta(x + y) = \Delta((m_x + e_x) + (m_y + e_y)) = \Delta(e_x + e_y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

The last step above is beyond the scope of this book. Its proof can be found at this link: `https://en.wikipedia.org/wiki/Sum_of_normally_distributed_random_variables`.

[4] The standard deviation of xy is

$$\Delta(xy) = \Delta((m_x + e_x)(m_y + e_y)) \approx \Delta(m_x m_y + m_x e_y + m_y e_x) = \Delta(m_x e_y + m_y e_x).$$

Applying the rule for a sum, we get $\Delta(xy) = \sqrt{(m_x\Delta y)^2 + (m_y\Delta x)^2}$.

[5] For rule of power: Let $x = m + e$, where m is the unknown true value of the measured quantity and e is a zero-mean Gaussian variable with standard deviation $\Delta e = \Delta x \ll m$. Then, the standard deviation of x^a is

$$\Delta(x^a) = \Delta((m + e)^a) = \Delta(m^a + am^{a-1}e + \ldots) \approx \Delta(m^a + am^{a-1}e).$$

Because m is a constant, we can write $\Delta(x^a) = \Delta(am^{a-1}e) = |a|m^{a-1}\Delta e = |a|m^{a-1}\Delta x$.

Using the rule for a power, we get that

$$\Delta(T^{-2}) = 2T^{-3}\Delta T = 0.025\ \mathrm{s}^{-2}.$$

Using the rule for a product along with the value we just obtained for $\Delta(T^{-2})$, we get that

$$\begin{aligned}\Delta(LT^{-2}) &= \sqrt{(L\Delta(T^{-2}))^2 + (T^{-2}\Delta L)^2}\\ &= \sqrt{(0.250\ \mathrm{s}^{-2})^2(0.05\ \mathrm{m})^2 + (1.00\ \mathrm{m})^2(0.025\ \mathrm{s}^{-2})^2}\\ &= 0.028\ \mathrm{m/s}^2.\end{aligned}$$

Multiplying this by the scalar $4\pi^2$ in the formula for g yields a final uncertainty of $\Delta g = 1.1\ \mathrm{m/s}^2$, so our answer is D.

2018.13. Let $F = 7300$ N, $d = 2.54$ cm, $r = d/2 = 1.27$ cm, and $\theta = 1.50°$. The stress can be straightforwardly written as

$$\frac{F}{A} = \frac{F}{\pi r^2}.$$

Also, using trigonometry, the cable increases in length by a proportion of $1/\cos\theta$, so the strain is

$$\frac{\Delta L}{L} = \frac{1}{\cos\theta} - 1.$$

Finally, the Young's modulus is

$$\begin{aligned}Y &= \frac{F/A}{\Delta L/L}\\ &= \frac{F}{\pi r^2}\left(\frac{1}{\cos\theta} - 1\right)^{-1}\\ &= \frac{7300\ \mathrm{N}}{\pi(0.0127\ \mathrm{m})^2}\left(\frac{1}{\cos 1.50°} - 1\right)^{-1}\\ &= \frac{1.44\times 10^7\ \mathrm{N/m}^2}{3.43\times 10^{-4}}\\ &= 4.2\times 10^{10}\ \mathrm{N/m}^2.\end{aligned}$$

Therefore, the answer is E.

2018.14. Analogous to a simple pendulum, the period T is proportional to the square root of the moment of inertia about the axis of rotation I. When pushed to the left, the moment of inertia is $3MR^2$ due to all of the masses having moment arm R, but when pushed into the page, the moment of inertia is only MR^2 as two masses are on the axis of rotation. Thus, the ratio of moments of inertia is 3, so the ratio of periods is $\sqrt{3}$, and the answer is C.

2018.15. By the virial theorem[6], the total energy of the satellite in orbit is equal to the negative of the average kinetic energy. Thus, the answer is A.

Alternatively, we can derive this fact using the orbital velocity formula. For an object of mass m in circular orbit of radius R about a larger object of mass $M \gg m$, we have that

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot \frac{GM}{R} = \frac{1}{2} \cdot \frac{GMm}{R} = -\frac{1}{2}U.$$

Thus, the total energy $K + U$ is equal to $-K$, as before, so the answer is A.

2018.16. This problem has two parts, testing knowledge of the centrifugal and Coriolis pseudo forces, respectively. First, the acceleration of the astronaut is directed away from the axis and toward the floor due to the centrifugal force, which rules out answer choices D and E.

The second part of the problem is a little trickier. If you're comfortable with the Coriolis force, this is as easy as a hand rule to compute the direction of $\vec{v} \times \vec{\omega}$, but otherwise, it requires careful thought. We will work in the inertial reference frame of the space station. At the instant the astronaut jumps, we draw a dot on the floor below him.

Now, we track the paths of the astronaut and the dot. Both move at a constant speed, but the astronaut takes a straight-line path, which is a chord in the circular cross-section of the space station, while the dot's path is an arc. Since a chord subtends a larger angle than an arc of equal length, the astronaut travels further than and must land in front of the dot. Thus, the answer is B.

2018.17. This problem can be tricky because it can take a couple of careful reads to understand what the graphs in the answer choices are really trying to say. Essentially, the graphs show a side-view snapshot of the scenario.

To solve this problem, first consider the simpler case where the helicopter does not turn, rather staying at constant velocity the entire time. Then since the stream of sand moves at the same velocity as the helicopter when it drops, it appears as a vertical line below the helicopter. In the scenario of the problem when the helicopter turns, it should instead have two disconnected vertical lines beneath it, so the answer is D.

2018.18. Here, the problem statement gives you the period T of the pendulum as a red herring, and you should not fall into the trap of trying to calculate the period via a messy integral. Instead, due to conservation of mechanical

[6] See `https://en.wikipedia.org/wiki/Virial_theorem`.

energy, note that the kinetic energy of the mass at the bottom of the pendulum is $2mg\ell$ greater than that at the top, so the centripetal force is increased by

$$\Delta F_c = m\Delta a_c = \frac{\Delta(mv^2)}{\ell} = \frac{2\Delta K}{\ell} = \frac{4mg\ell}{\ell} = 4mg.$$

Also, gravity on the mass acts away from the center at the bottom as opposed to towards the center at the top, so the total difference in tensions is $4mg + 2mg = 6mg$. Therefore, the answer is C.

Note: Alternatively, we can explicitly write formulas for the tension as follows. Let v_0 and v_1 be the velocities of the mass at the bottom and top of its circle, respectively, and let T_0 and T_1 be the corresponding tensions in the rod. Then,

$$T_0 = \frac{mv_0^2}{\ell} + mg \qquad \text{and} \qquad T_1 = \frac{mv_1^2}{\ell} - mg.$$

However, from conservation of energy, we know that $v_0^2 = v_1^2 + 4gl$. Combining this with the equations above for tension yields $\Delta T = T_0 - T_1 = 4mg$, so the answer is C.

2018.19. The key to solving this problem is to consider ratios carefully, as it can be easy to be confused by the wording of the statement. Also, make sure to use momenta here, as the collisions of raindrops with the ground are certainly inelastic. Consider a flat area A, and let ρ be the density of water and L be the volumetric flow rate of rain over this area. Over a small time interval Δt, we can equate impulse and momentum to get

$$PA\Delta t = J = \Delta p = \rho L v \Delta t,$$

$$P = \frac{\rho L v}{A}.$$

Also, note that the numeric rate at which raindrops hit the surface of area A is given by nAv. If we multiply this by the volume of each raindrop, this gives us the volumetric rate

$$L = \frac{4}{3}\pi \cdot nAvr^3.$$

Combining this with the previous expression for pressure, we can derive the formula

$$P = \frac{4}{3}\pi\rho \cdot nv^2r^3.$$

With this formula in mind, note that the problem asks us what happens when n is doubled, r is halved, and v is halved. The new pressure is equal to

$$P_0(2)(0.5)^2(0.5)^3 = P_0/16.$$

Thus, the answer is E.

2018.20. The cut makes no difference. To see why, you can imagine attaching the half springs end-to-end while you're stretching them from a resting position. This results in a scenario identical to the original one, with a whole spring of double the length. Thus, the answer is C.

Alternatively, you can use the fact that the constant k for springs of the same composition and width is inversely proportional to the spring's resting length. When compared to the original, each of the two half-springs have double the spring constant $k' = 2k_0$, but their unstretched length $x' = x_0/2$ is half of the original. Then, the potential energy of a single half-spring is

$$U' = \frac{1}{2}k'x'^2 = \frac{1}{2}(2k_0)(x_0/2)^2 = \frac{1}{2}U_0.$$

Then, the total potential energy stored by the two half-springs is $2U' = U_0$, so the answer is C.

2018.21. In the time interval between when the ball is first placed on the ground ($t = 0$), and when the ball begins rolling without slipping, the forces acting on the ball are gravity, the normal force with the ground, and kinetic friction. The former two of these forces cancel each other out, so the net force is that of kinetic friction, given by $F = \mu mg$.

Using $F = ma$, the translational acceleration of the ball is then $a = -\mu g$. To calculate the angular acceleration, the moment of inertia for a hollow spherical shell of mass m and radius r is given by the formula

$$I = \frac{2}{3}mr^2.$$

Then, since the torque is $\tau = Fd = \mu mgr$, the angular acceleration is

$$\tau = I\alpha \quad \Rightarrow \quad \alpha = \frac{\tau}{I} = \frac{\mu mgr}{\frac{2}{3}mr^2} = \frac{3}{2}\mu g/r.$$

Since the magnitude and direction of the net force is constant, the translational and angular velocities at time t are linear, given by $v(t) = v_0 - \mu gt$ and $\alpha(t) = \frac{3}{2}\mu gt/r$. The problem asks us to find the first time when $v = r\alpha$, so we can solve the equation to get

$$v_0 - \mu gt = \frac{3}{2}\mu gt \quad \Rightarrow \quad t = (2/5)v_0/\mu g \quad \Rightarrow \quad \text{the answer is A.}$$

2018.22. In general, the force of buoyancy that acts to keep an object afloat is equal in magnitude to the weight of the displaced fluid. This is known as Archimedes' Principle. We will use this to show that the difference in water levels inside and outside the boat remains constant.

For this problem, assuming a sufficiently small hole, the boat will always move to achieve equilibrium when it has a given amount of water inside of it. To analyze this, the forces of gravity and buoyancy acting on the boat can be divided into two parts. First is the weight of the boat F_g, and second is the combination of the weight of the air inside the boat (but under sea level) and the force of buoyancy on the boat, which we will call F_p.

If we assume that the solid hull of the boat is thin (and thus displaces little water in itself), then F_p is approximately proportional to the volume of air inside the boat that is below water level. This means that $F_p \propto \Delta h$, where Δh is the difference in water levels inside and outside the boat. However, since the weight of the boat F_g remains constant, static equilibrium means that F_p and Δh similarly remain constant.

Thus, the pressure differential between the inside and outside of the hole remains the same as the ship sinks, so the leak rate through the hole should remain constant. The answer is A.

2018.23. During the actual contest, this problem is a great example of using the answer choices to your advantage, since each of the graphs has a very different shape. Problems very similar to this one have previously appeared on $F = ma$ exams (c.f. Problem 2015.22). General strategies for approaching this kind of problem include

- Looking at continuity, as most graphs you see will be continuous.
- Examining limiting cases, such as the smallest or largest values of the independent variable.
- Looking at the convexity of the graph.
- Analyzing the problem for points at which the graph could possibly change (for example, the change from static to kinetic friction).

In this case, noting the difference between static and kinetic friction indicates that answer choice C could be correct, as it is continuous but consists of two smooth pieces. However, limiting cases let us be more certain of this answer. At ramp angle $\theta = 0°$, the ball does not move, so the angular acceleration is zero. Similarly, at ramp angle $\theta = 90°$, the normal force (and thus the force of friction) on the ball is zero, so the angular acceleration is again zero. Only one graph has zeros at these two values, so the answer is C.

2018.24. It is a well known fact that for simple pendulums, as the maximum angular displacement θ_0 is increased to π, the small-angle approximation $\sin\theta \approx \theta$ diverges, and the period increases since $\sin\theta < \theta$.

Another way of seeing this is that when $\theta_0 = \pi$, the mass does not move at all, as it is balanced perfectly in an (unstable) equilibrium directly above the axle. Then the period for this angle should be infinite, and assuming continuity, there should be a vertical asymptote in the graph of the period at $\theta_0 = \pi$. In turn, the period of motion should initially increase.

However, when kinetic energy K is increased to exceed the potential energy difference between the top and bottom of the circle, this model no longer works. Instead, the mass spins around the circle non-uniformly, but with no "peaks," i.e., moments when its velocity is zero. When K is orders of magnitude larger than the potential energy difference, this motion is approximately uniform circular, so the period should decrease inversely proportional to the speed, approaching a horizontal asymptote of 0. Therefore, the period initially increases, then decreases, so the answer is E.

2018.25. Similar to Problem 12 in this exam, this problem deals with error propagation. Intuitively, it might at first seem that Method 1 has the lowest uncertainty, as Bob's measurement is less precise. However, Bob's measurement still matters because it is independent from Alice's. It is a rule of good experimental design to combine the results of independent repeated trials, and Alice and Bob's case is no exception to this.

If you're still skeptical, consider the less extreme case where Alice's measurement is 1.013 ± 0.008 s, while Bob's measurement is 0.997 ± 0.0081 s. Here, it really wouldn't make sense to throw out Bob's measurement just because it is slightly less precise than Alice's.

What does this mean for the current situation? The key is that the uncertainties of independent measurements add geometrically when the measurements are summed. Specifically, for the case of two measurements x_1 and x_2 of uncertainty Δx_1 and Δx_2, the uncertainty of their sum is

$$\Delta(x_1 + x_2) = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2}.$$

Letting Alice's uncertainty be Δa and Bob's uncertainty be Δb, the uncertainties of the three measurements are, respectively:

1. $\Delta a = 0.008$ s,
2. $\Delta(a + b)/2 = \sqrt{(\Delta a)^2 + (\Delta b)^2}/2 = 0.0089$ s, and
3. $\Delta(4a + b)/5 = \sqrt{(4\Delta a)^2 + (\Delta b)^2}/5 = 0.0072$ s.

Therefore, the answer is B.

Note: In general, when we have multiple independent measurements with the same uncertainty, we can get the most precise result by averaging them equally. However, when multiple measurements have differing uncertainties, the best weights are inversely proportional to the squares of the uncertainties, as demonstrated by Method 3 in this problem.

Showing this is a straightforward application of Lagrange multipliers. Another, perhaps more elegant, algebraic approach is to use the Cauchy-Schwarz Inequality. This yields, for weights w_i summing to 1 and measurements x_i,

$$\sqrt{\left(\sum_i (w_i \Delta x_i)^2\right)\left(\sum_i (1/\Delta x_i)^2\right)} \geq \sum_i w_i = 1.$$

The minimum uncertainty is achieved at the equality case, when the two sequences are multiples of each other, or

$$w_i \Delta x_i \propto 1/\Delta x_i \quad \Rightarrow \quad w_i \propto \frac{1}{\Delta x_i^2}.$$

2018 $F = ma$ Exam B

ToC

AAPT UNITED STATES PHYSICS TEAM AIP 2018

2018 $F = ma$ Contest

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

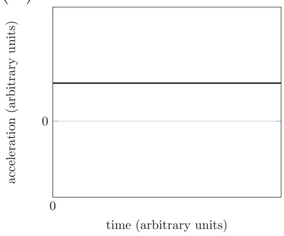
DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Your answer to each question must be marked on the optical mark answer sheet.
- Select the single answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- This test contains 25 multiple choice questions. Your answer to each question must be marked on the optical mark answer sheet that accompanies the test. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 20, 2018.**
- The question booklet and answer sheet will be collected at the end of this exam. You may not use scratch paper.

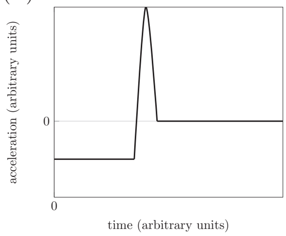
DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

1. A large lump of clay is dropped off of a wall and lands on the ground. Which graph best represents the acceleration of the center of mass of the clay as a function of time?

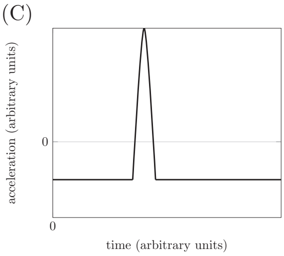
(A)



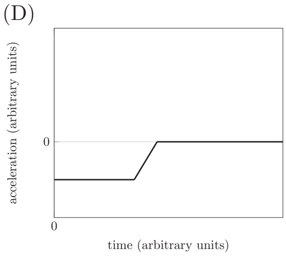
(B)



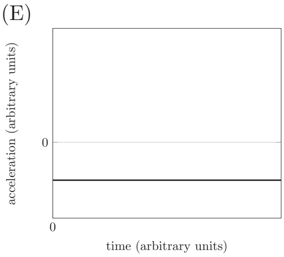
(C)



(D)



(E)



2. A uniform block of mass 10 kg is released at rest from the top of an incline with length 10 m and inclination 30°, and slides to the bottom. The coefficients of static and kinetic friction are $\mu_s = \mu_k = 0.1$. How much energy is dissipated due to friction?

   (A) 0 J
   (B) 22 J
   (C) 43 J
   (D) 87 J
   (E) 164 J

3. A 3.0 kg mass moving at 40 m/s to the right collides with and sticks to a 2.0 kg mass traveling at 20 m/s to the right. After the collision, the kinetic energy of the system is closest to

   (A) 600 J
   (B) 1200 J
   (C) 2600 J
   (D) 2800 J
   (E) 3400 J

4. A basketball is released from rest and bounces on the ground. Considering only the ball just before and just after the bounce, which of the following statements must be true?

   (A) The momentum and the total energy of the ball are conserved.
   (B) The momentum of the ball is conserved, but not the kinetic energy.
   (C) The total energy of the ball is conserved, but not the momentum.
   (D) The kinetic energy of the ball is conserved, but not the momentum.
   (E) Neither the kinetic energy of the ball nor the momentum is conserved.

5. The hard disk in a computer will spin up to speed within 10 rotations, but when turned off will spin through 50 rotations before coming to a stop. Assuming the hard disk has constant angular acceleration α_1 and angular deceleration α_2, the ratio α_1/α_2 is

   (A) $1/5$
   (B) $1/\sqrt{5}$
   (C) $\sqrt{5}$
   (D) 5
   (E) 25

6. A massless beam of length L is fixed on one end. A downward force F is applied to the free end of the beam, deflecting the beam downward by a distance x. The deflection x is linear in F and is inversely proportional to the cross-section moment I, which has units m^4. The deflection is also dependent on Young's modulus E, which has units $\mathrm{N/m}^2$. Then x depends on L according to

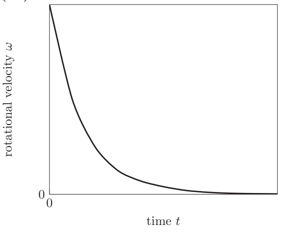
   (A) $x \propto \sqrt{L}$
   (B) $x \propto L$
   (C) $x \propto L^2$
   (D) $x \propto L^3$
   (E) $x \propto L^4$

7. A pendulum of length L oscillates inside a box. A person picks up the box and gently shakes it vertically with frequency ω and a fixed amplitude for a fixed time. To maximize the final amplitude of the pendulum, ω should satisfy

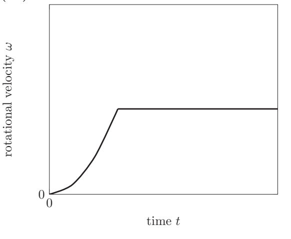
   (A) $\omega = \sqrt{4g/L}$
   (B) $\omega = \sqrt{2g/L}$
   (C) $\omega = \sqrt{g/L}$
   (D) $\omega = \sqrt{g/4L}$
   (E) there will be no significant effect on the pendulum amplitude for any value of ω

8. The coefficients of static and kinetic friction between a ball and a horizontal plane are $\mu_s = \mu_k = \mu$. The ball is given a horizontal speed but no rotational velocity about its center of mass. Which of the following graphs best shows the rotational velocity of the ball about its center of mass as a function of time?

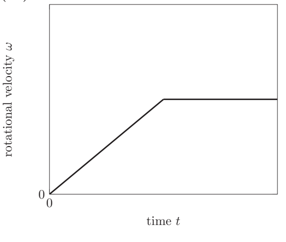
(A)



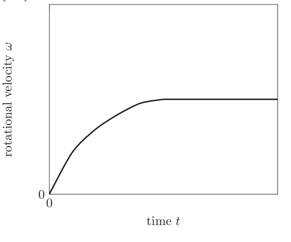
(B)



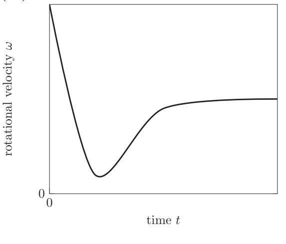
(C)



(D)



(E)



9. A 3.0 kg ball moving at 10 m/s east collides elastically with a 2.0 kg ball moving 15 m/s west. Which of the following statements could be true after the collision?

   (A) The two balls are both moving directly east.
   (B) The 3.0 kg ball moves directly west at 15 m/s.
   (C) The 2.0 kg ball moves directly north at 10 m/s.
   (D) The 3.0 kg ball is at rest.
   (E) The 2.0 kg ball moves directly south at 15 m/s.

10. A balloon filled with air submerged in water at a depth h experiences a buoyant force B_0. The balloon is moved to a depth of $2h$, where it experiences a buoyant force B. Assuming the water is incompressible and the balloon and air are compressible, the buoyant force B satisfies

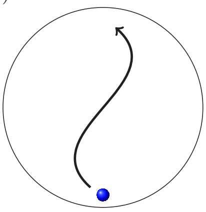
    (A) $B \geq 2B_0$
    (B) $B_0 < B < 2B_0$
    (C) $B = B_0$
    (D) $B < B_0$
    (E) it depends on the compressibility of the balloon and air

11. A circle of rope is spinning in outer space with an angular velocity ω_0. Transverse waves on the rope have speed v_0, as measured in a rotating reference frame where the rope is at rest. If the angular velocity of the rope is doubled, the new speed of transverse waves, as measured in a rotating reference frame where the rope is at rest, will be

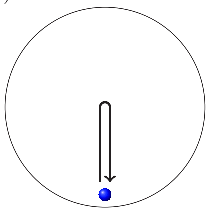
    (A) v_0
    (B) $\sqrt{2}v_0$
    (C) $2v_0$
    (D) $4v_0$
    (E) $8v_0$

12. A child in a circular, rotating space station tosses a ball in such a way so that once the station has rotated through one half rotation, the child catches the ball. From the child's point of view, which plot shows the trajectory of the ball? The child is at the bottom of the space station in the diagrams below, but only the initial location of the ball is shown.

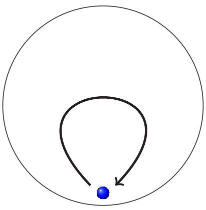
(A)



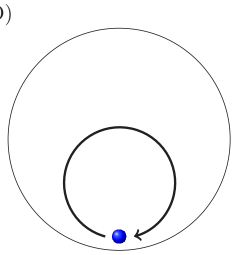
(B)



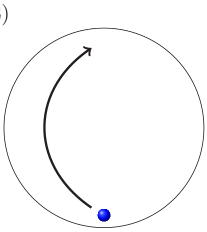
(C)



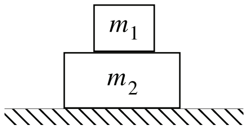
(D)



(E)

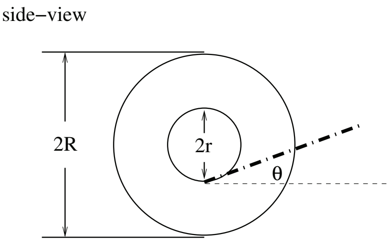


13. Two blocks of masses $m_1 = 2.0$ kg and $m_2 = 1.0$ kg are stacked together on top of a frictionless table as shown. The coefficient of static friction between the blocks is $\mu_s = 0.20$. What is the minimum horizontal force that must be applied to the top block to make it slide across the bottom block?



(A) 4.0 N
(B) 6.0 N
(C) 8.0 N
(D) 12.0 N
(E) The top block will not slide across the bottom block

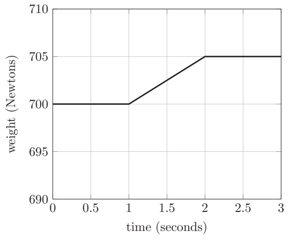
14. A spool is made of a cylinder with a thin disc attached to either end of the cylinder, as shown. The cylinder has radius $r = 0.75$ cm and the discs each have radius $R = 1.00$ cm. A string is attached to the cylinder and wound around the cylinder a few times. At what angle above the horizontal can the string be pulled so that the spool will slip without rotating?



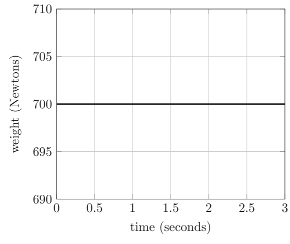
(A) 31.2°
(B) 41.4°
(C) 54.0°
(D) 60.8°
(E) 81.5°

15. You are standing on a weight scale that reads 700 Newtons while holding a large physics textbook that is originally at rest. At time $t = 1$ seconds you begin moving the textbook upward so that by time $t = 2$ seconds the textbook is now half a meter higher and once again at rest. Which of the following graphs best illustrates how the reading on the scale might vary with time?

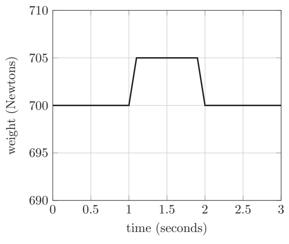
(A)



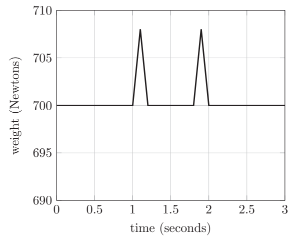
(B)



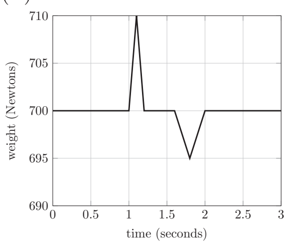
(C)



(D)



(E)

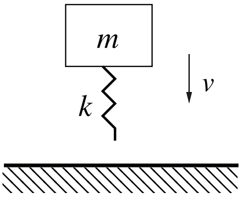


16. A plane can fly by tilting the trailing edge of the wings downward by a small angle θ, called the angle of attack. In still air, a plane with ground speed v will have a lift force proportional to $v^2\theta$ and a drag force is proportional to v^2.

    Consider a plane initially in level flight in still air with constant ground speed v. If the plane enters a region with a tailwind with speed $w < v$ (the wind is blowing in the same direction that the plane wants to fly), how must the engine power and the angle of attack change for the plane to maintain level flight at the same ground speed?

    (A) The engine power decreases and the angle of attack decreases
    (B) The engine power decreases and the angle of attack stays the same
    (C) The engine power decreases and the angle of attack increases
    (D) The engine power increases and the angle of attack decreases
    (E) The engine power increases and the angle of attack increases

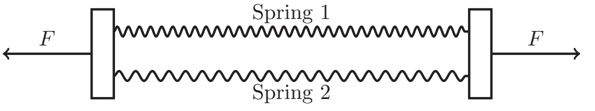
17. A pogo stick is modeled as a massless spring of spring constant k attached to the bottom of a block of mass m. The pogo stick is dropped with the spring pointing downward and hits the ground with speed v. At the moment of the collision, the free end of the spring sticks permanently to the ground.



    During the subsequent oscillations, the maximum speed of the block is

    (A) v
    (B) $v + 2mg^2/kv$
    (C) $v + mg^2/kv$
    (D) $\sqrt{v^2 + 2mg^2/k}$
    (E) $\sqrt{v^2 + mg^2/k}$

18. A spring of relaxed length ℓ_1 and spring constant k_1 is placed 'in parallel' with a spring of relaxed length ℓ_2 and spring constant k_2. A force F is applied to each end.



The combination of the springs acts like a single spring with spring constant k and relaxed length ℓ where

(A) $k = k_1 + k_2$ and $\ell = \ell_1\ell_2/(\ell_1 + \ell_2)$
(B) $k = k_1 + k_2$ and $\ell = (\ell_1 k_1 + \ell_2 k_2)/(k_1 + k_2)$
(C) $k = k_1 + k_2$ and $\ell = (\ell_1 k_2 + \ell_2 k_1)/(k_1 + k_2)$
(D) $k = (\ell_1 k_1 + \ell_2 k_2)/(\ell_1 + \ell_2)$ and $\ell = (\ell_1 k_1 + \ell_2 k_2)/(k_1 + k_2)$
(E) $k = (\ell_2 k_1 + \ell_1 k_2)/(\ell_1 + \ell_2)$ and $\ell = (\ell_1 k_2 + \ell_2 k_1)/(k_1 + k_2)$

19. In an experiment to determine the speed of sound, a student measured the distance that a sound wave traveled to be 75.0 ± 2.0 cm, and found the time it took the sound wave to travel this distance to be 2.15 ± 0.10 ms. Assume the uncertainties are Gaussian. The computed speed of sound should be recorded as

(A) 348.8 ± 0.5 m/s
(B) 348.8 ± 0.8 m/s
(C) 349 ± 8 m/s
(D) 349 ± 15 m/s
(E) 349 ± 19 m/s

20. A massive, uniform, flexible string of length L is placed on a horizontal table of length $L/3$ that has a coefficient of friction $\mu_s = 1/7$, so equal lengths $L/3$ of string hang freely from both sides of the table. The string passes over the edges of the table on smooth, frictionless, curved surfaces.

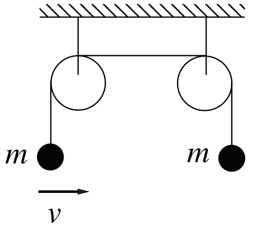
Now suppose that one of the hanging ends of the string is pulled a distance x downward, then released at rest. Neither end of the string ever touches the ground in this problem. The maximum value of x so that the string does not slip off of the table is

(A) $L/42$
(B) $L/21$
(C) $L/14$
(D) $2L/21$
(E) $3L/14$

21. A uniform bar of length L and mass M is supported by a fixed pivot a distance x from its center. The bar is released from rest from a horizontal position. The period of the resulting oscillations is minimal when

    (A) $x = L/2$
    (B) $x = L/2\sqrt{3}$
    (C) $x = L/4$
    (D) $x = L/4\sqrt{3}$
    (E) $x = L/12$

22. Two particles of mass m are connected by pulleys as shown.



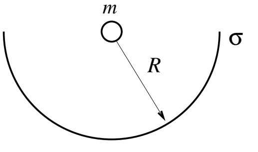
The mass on the left is given a small horizontal velocity, and oscillates back and forth. The mass on the right

    (A) remains at rest
    (B) oscillates vertically, and with a net upward motion
    (C) oscillates vertically, and with a net downward motion
    (D) oscillates vertically, with no net motion
    (E) oscillates horizontally, with no net motion

23. Two particles with mass m_1 and m_2 are connected by a massless rigid rod of length L and placed on a horizontal frictionless table. At time $t = 0$, the first mass receives an impulse perpendicular to the rod, giving it speed v. At this moment, the second mass is at rest. The next time the second mass is at rest is

    (A) $t = 2\pi L/v$
    (B) $t = \pi(m_1 + m_2)L/m_2 v$
    (C) $t = 2\pi m_2 L/(m_1 + m_2)v$
    (D) $t = 2\pi m_1 m_2 L/(m_1 + m_2)^2 v$
    (E) $t = 2\pi m_1 L/(m_1 + m_2)v$

24. A particle of mass m is placed at the center of a hemispherical shell of radius R and mass density σ, where σ has dimensions of kg/m^2.



The gravitational force of the shell on the particle is

(A) $(1/3)(\pi G m \sigma)$
(B) $(2/3)(\pi G m \sigma)$
(C) $(1/\sqrt{2})(\pi G m \sigma)$
(D) $(3/4)(\pi G m \sigma)$
(E) $\pi G m \sigma$

25. A student is measuring the surface area of a cylindrical wire. The student measures the radius of the wire to be 1 ± 0.1 cm using a ruler and the length of the wire to be 1.00 ± 0.01 m using a meter-stick. The precision of their result can be increased in several ways.

1: Upgrade the ruler to a caliper with uncertainty 0.01 cm.
2: Upgrade the meter-stick to a tape measure with uncertainty 0.001 m.
3: Repeat the measurement independently ten times and average the results.

How do the resulting uncertainties of the measurements compare?

(A) Method 3 has the lowest uncertainty, while methods 1 and 2 have the same uncertainty
(B) Method 3 has the highest uncertainty, while methods 1 and 2 have the same uncertainty
(C) Method 1 has the highest uncertainty, and method 2 has the lowest
(D) Method 2 has the highest uncertainty, and method 1 has the lowest
(E) Method 2 has the highest uncertainty, and method 3 has the lowest

Answers, Problem Difficulty, and Topics

ToC

2018-B.1	B	⋆	projectile motion
2018-B.2	D	⋆	kinetic friction
2018-B.3	C	⋆	collisions, kinetic energy
2018-B.4	E	⋆	collisions, kinetic energy
2018-B.5	D	⋆	circular motion
2018-B.6	D	⋆	dimensional analysis
2018-B.7	A	⋆ ⋆	simple pendulum, oscillations
2018-B.8	C	⋆	kinetic friction
2018-B.9	E	⋆ ⋆	collisions
2018-B.10	D	⋆ ⋆	Archimedes' Principle
2018-B.11	C	⋆ ⋆	waves, centripetal force, tension
2018-B.12	C	⋆ ⋆	rotating reference frame
2018-B.13	D	⋆	static friction
2018-B.14	B	⋆ ⋆	rolling motion,
2018-B.15	E	⋆	forces
2018-B.16	C	⋆	power, air friction
2018-B.17	E	⋆ ⋆	springs, energy
2018-B.18	B	⋆ ⋆	springs, Hooke's Law
2018-B.19	E	⋆ ⋆	measurement error analysis
2018-B.20	A	⋆ ⋆ ⋆	static friction, tension
2018-B.21	B	⋆ ⋆ ⋆	physical pendulum, oscillations
2018-B.22	B	⋆ ⋆ ⋆	tension, Atwood machine, oscillations
2018-B.23	A	⋆ ⋆	circular motion
2018-B.24	E	⋆ ⋆ ⋆	gravitation, pressure
2018-B.25	D	⋆ ⋆	measurement error analysis

Solutions for Exam B

ToC

2018.1. The motion of the object can be described in three phases.

1. First, while the clay is in free fall, it has a constant negative acceleration of $a = -g = -10\ \mathrm{m/s}^2$, which should be indicated by a horizontal line below the time axis. This means that answer choice A is incorrect.

2. When the object lands, it experiences a sudden upwards acceleration due to the collision with the ground, indicated by a large vertical spike in the graph. This observation narrows our options down to answer choices B and C.

3. Afterwards, the object on the ground remains at rest, having constant, zero acceleration. In the graph, this should result in a horizontal line at the time axis.

Therefore, the answer is B.

2018.2. The work on the object due to friction is given by the force of friction F_f multiplied by distance over which it is applied. If we denote the length of the incline by d, we get that

$$\begin{aligned} W &= F_f \cdot d \\ &= -\mu_k N \cdot d \\ &= -\mu_k mg\cos\theta \cdot d \\ &= -(0.1)(10\ \mathrm{kg})(10\ \mathrm{m/s}^2)(\cos 30^\circ)(10\ \mathrm{m}) \\ &= -50\sqrt{3}\ \mathrm{J}. \end{aligned}$$

This works out to be about 87 J of energy dissipated, so the answer is D.

2018.3. Since the objects stick together, the collision in this problem is perfectly inelastic. First, we calculate the initial momentum of the system as the sum of the momenta of its parts, or

$$p_i = m_1 v_1 + m_2 v_2 = (3.0\ \mathrm{kg})(40\ \mathrm{m/s}) + (2.0\ \mathrm{kg})(20\ \mathrm{m/s}) = 160\ \mathrm{kg\ m/s}.$$

However, by conservation of momentum, this initial momentum p_i should equal to the final momentum after collision p_f. Since the masses after collision combine into a single object of mass $m_1 + m_2 = 5.0$ kg, the kinetic energy can be calculated as

$$K_f = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{p_f^2}{2m} = \frac{(160\ \mathrm{kg\ m/s})^2}{2(5.0\ \mathrm{kg})} = 2560\ \mathrm{J}.$$

Thus, the answer is C.

2018.4. The kinetic energy of the ball is not conserved, as when the ball hits the floor, some of its mechanical energy is converted into sound and heat energy. This is why you can hear the sound of a ball bouncing, and it is also why colliding steel balls can burn a piece of paper between them[7]. This energy loss explains why the height of the peak of a ball's motion decreases with every bounce.

Momentum is also not conserved, as the ball receives an external force from the ground. You can also see this because the ball's velocity changes direction. Therefore, the answer is E.

2018.5. From the rotational kinematic formulas, we know that $\omega_f^2 - \omega_i^2 = 2\alpha\theta$ for a process with constant angular acceleration α. In this problem, the acceleration and deceleration parts have the same difference between initial and final squared angular velocities, so the magnitude of $\alpha\theta$ is the same. Then, $\theta_1/\theta_2 = 10/50 = 1/5$, so $\alpha_1/\alpha_2 = 5$, and the answer is D.

2018.6. This is an exercise in dimensional analysis. First, notice that all quantities are in length units, except for the newtons in the units of Young's modulus E and force F, so these force units must cancel each other out. Since the deflection x is linear in F, it must therefore be inversely proportional to E. If we let the exponent of L in the formula for x be γ, we have that

$$x = \frac{F}{IE}L^\gamma \quad \Rightarrow \quad L^\gamma = \frac{IEx}{F}.$$

The dimensions of this are

$$\left[\frac{IEx}{F}\right] = \frac{\mathrm{m}^4 \cdot \mathrm{N} \cdot m^{-2} \cdot \mathrm{m}}{\mathrm{N}} = \mathrm{m}^3.$$

Thus, $\gamma = 3$, and the answer is D.

2018.7. First, note that any acceleration of the box due to the person's shaking is manifested in the box's frame as a pseudo-force in the opposite direction. In effect, this leads to a gravitational acceleration g that oscillates between higher and lower than normal. Note, however, that since the shaking is "gentle," we have that $a \ll g$, so the frequency $\omega \approx \sqrt{g/L}$ of the pendulum should remain approximately the same.

With this approximation of the pendulum's period staying relatively constant, we can now approach the problem from a logical point of view. For the amplitude (maximum angle) of the pendulum to be high, we want gravity to be

[7] See here for a video demonstration: `https://youtu.be/COWOv8aOoSU`.

greater when the pendulum is moving downwards, so it gains more speed, and weaker when it is moving upwards, so that it reaches a greater height. Each half-period of the pendulum's swing should then correspond to one complete period of the box shaking. Thus, the frequency of the box shaking should be double that of the pendulum, so the answer is A.

2018.8. First, note that we can immediately eliminate answers A and E, as the problem states that the angular velocity at time zero is zero. Initially, as the ball slides on the plane, kinetic friction at the contact point should exert a constant force of magnitude $F_f = \mu_k mg$, which acts to give the ball a constant torque. This should correspond to a linear increase in the ball's angular velocity.

However, when the ball begins rolling without slipping at $\omega = v/r$, it no longer has a force of friction with the ground, as the instantaneous velocity of the mass at the point of contact is zero. The graph should then level out to a horizontal line, showing a constant angular velocity. Thus, the answer is C.

2018.9. Since the collision is elastic, momentum (in both dimensions) and kinetic energy are conserved. Note that the initial momentum is

$$p = m_1 v_1 + m_2 v_2 = (3.0\text{ kg})(10\text{ m/s}) + (2.0\text{ kg})(-15\text{ m/s}) = 0.$$

The final momentum should then also be zero. This means that after collision, the two balls should be moving in opposite directions along the same axis passing through the collision point such that their momenta are equal and opposite. We will work in this axis.

Since the balls' momenta are equal in magnitude, let the speed of the 3.0 kg and 2.0 kg balls be $2v$ and $3v$, respectively. By conservation of energy, we have

$$\frac{m_1 u_1^2}{2} + \frac{m_2 u_2^2}{2} = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2},$$

$$\frac{(3.0\text{ kg})(10\text{ m/s})^2}{2} + \frac{(2.0\text{ kg})(15\text{ m/s})^2}{2} = \frac{(3.0\text{ kg})(2v)^2}{2} + \frac{(2.0\text{ kg})(3v)^2}{2}.$$

It is easy to solve this equation by observation, as initial velocities just equal the final velocities. Alternatively, we could simplify and end up with

$$375\text{ J} = (15.0\text{ kg})v^2 \quad \Rightarrow \quad v = 5.0\text{ m/s}.$$

After collision, the 3.0 kg ball should thus be moving at a speed of $2v = 10$ m/s, while the 2.0 kg ball should be moving at $3v = 15$ m/s in the opposite direction. This rules out A for direction reasons and B, C, D for speed reasons, so the answer is E.

2018.10. At a depth of $2h$, the gauge pressure on the balloon is doubled. Because the air in the balloon is compressible, its volume will decrease proportionally to the pressure. Since less water is displaced by a smaller volume of air, the buoyant force will be lower, so the answer is D.

2018.11. The centripetal force on a small section of rope with mass m is $F_c = mr\omega^2$. This force is provided by and is proportional to the tension in the circle of rope[8]. Since wave speed $v = \sqrt{T/\mu}$, where μ is mass density (constant in this scenario), we have

$$v \propto \sqrt{T} \propto \sqrt{F_c} \propto \omega \quad \Rightarrow \quad \text{the answer is C.}$$

2018.12. Since the child both throws and catches the ball, the path of the ball in the child's reference frame must begin and end at the child's position, narrowing the answers down to B, C, and D. Next, we examine the initial velocity of the ball. Consider the inertial reference frame with the child at the South end of the circle at time of the initial throw. The ball's velocity is due North, while the child's velocity is East, so the initial velocity of the ball in the child's reference frame is Northwest[9]. Therefore, the answer is C.

2018.13. Consider the scenario below the threshold force, when the blocks do not slide. Then, we can treat the blocks together as a system, so their acceleration is

$$a = \frac{F}{m_1 + m_2}.$$

However, the only horizontal force acting on the bottom block is static friction F_s with the top block, so we must have that

$$F_s = m_2 a = \frac{m_2}{m_1 + m_2} F = \frac{F}{3}.$$

The normal force between the blocks is equal to magnitude to the force of gravity on m_1, i.e., $N = m_1 g$. We can then calculate the threshold for slipping as

$$F_s < \mu_s N \quad \Rightarrow \quad F < 3\mu_s m_1 g = 3(0.20)(2.0 \text{ kg})(10 \text{ m/s}) = 12 \text{ N}.$$

Thus, the answer is D.

[8] See Problem 2016.23 for a similar problem involving explicit calculation of this tension force in a circular rubber band.

[9] The directions here are actually arbitrary. The important observation is that the ball's initial velocity in the child's frame should neither be in the child's direction, as in D, or perpendicular to the child's direction, as in B.

2018.14. The trick to solving this problem is that the motion of any rolling rigid body can be described by a rotation with some angular velocity about the *contact point*, as well as a translational component. The advantage of this representation is that the gravity and friction acting on the object have zero torque along the contact-point axis, as the length of the moment arm is zero. For the spool to slip without rotating, the torque from the string also needs to be zero. In other words, the contact point of the spool with the ground should lie on the line of the string.

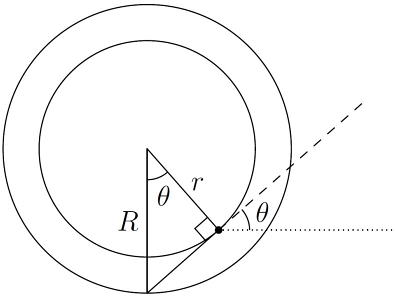


Figure 1: Illustration for Problem 2018.14.

As shown in Figure 1, if we draw the right triangle connecting the center of the spool, the tangency point of the string, and the contact point with the ground, we can use trigonometry to get that

$$\cos\theta = \frac{r}{R} \quad \Rightarrow \quad \theta = \arccos(r/R) = 41.4^\circ.$$

Thus, the answer is B.

2018.15. Consider the combined system of you and the textbook. The difference between the reading on the scale (the normal force) and the system's weight is the net force on the system. In particular, since the system begins and ends at rest, the weight reading on the scale should be the same before and after, ruling out answer choice A. This also means the net change in momentum (impulse) is zero, leaving us with answer choices B and E. Clearly, the force on the system cannot be a constant zero, as then the center of mass would remain at rest, so the answer is E.

2018.16. When the plane enters a tailwind while keeping the same ground velocity, its velocity in the reference frame of the air surrounding it will decrease ($v \mapsto v - w$). The lift force must remain equal in magnitude to the weight of the aircraft, so the angle of attack θ must increase with v^{-2}. However, the engine power, $P = F_{drag}v \propto v^3$, will decrease. Thus, the answer is C.

2018.17. At the instant that the pogo stick hits the ground, the spring is at its usual rest length ℓ, but the actual equilibrium length is $\ell - mg/k$ due to the gravitational potential on the block. The maximum kinetic energy is then the sum of the potential and kinetic energies at impact, or

$$\begin{aligned} K_{max} &= \frac{1}{2}mv^2 + \frac{1}{2}k(mg/k)^2 \\ &= \frac{1}{2}m(v^2 + mg^2/k). \end{aligned}$$

Then, by the formula for kinetic energy,

$$K_{max} = \frac{1}{2}mv_{max}^2 \quad \Rightarrow \quad v_{max} = \sqrt{v^2 + mg^2/k} \quad \Rightarrow \quad \text{the answer is E.}$$

2018.18. First, when springs are placed in parallel, the value of the spring constant k is summed. In order words, $k = k_1 + k_2$, which rules out answer choices D and E.

We now need to find the relaxed length ℓ of the two springs together. At this length, the system should be in static equilibrium, so

$$F_{net} = 0 = k_1(\ell - \ell_1) + k_2(\ell - \ell_2) \quad \Rightarrow \quad k_1\ell_1 + k_2\ell_2 = (k_1 + k_2)\ell.$$

Dividing both sides by $k_1 + k_2$ gives us the formula for the combined rest length, and the answer is B.

2018.19. We use relative uncertainties. When two measurements are multiplied or divided, their relative uncertainties add geometrically, giving

$$\frac{\Delta v}{v} \approx \sqrt{\left(\frac{\Delta s}{s}\right)^2 + \left(\frac{\Delta t}{t}\right)^2} = \sqrt{\left(\frac{2.0}{75.0}\right)^2 + \left(\frac{0.10}{2.15}\right)^2} = 5.4\%.$$

Using a first-order approximation, we estimate v to be the quotient of the measurements. Then the uncertainty is

$$\Delta v = \frac{\Delta v}{v}v \approx 5.4\% \cdot \frac{75.0 \text{ cm}}{2.15 \text{ ms}} = 19 \text{ m/s}.$$

Thus, the answer is E.

2018.20. Let the total mass of the string be m. First, we will list all the external forces contributing to the motion of the string. These are: gravity, the normal force, and static friction with the table. The normal force is responsible for changing the direction of the string's tension at its rounded corners, which holds the hanging ends of the string up. However, this force is on the smooth, frictionless edges of the table. The magnitude of the normal force on the string above the flat part of the table, with static friction, is $mg/3$.

Let T_1 be the tension in the string on the side of longer hanging end, and let T_2 be the tension on the shorter end. Since the string is in equilibrium, these forces must be equal to gravity on the hanging ends, so

$$T_1 = \left(\frac{1}{3} + \frac{x}{L}\right) mg, \quad T_2 = \left(\frac{1}{3} - \frac{x}{L}\right) mg.$$

Between the two ends of the table, the only external force acting parallel to the string (in the horizontal direction) is static friction, so this must account for the difference in tensions. Hence,

$$F_s = T_1 - T_2 = \frac{2x}{L} mg.$$

Since static friction is bounded by $F_s < \mu_s N = \frac{\mu_s}{3} mg$, this gives us an upper bound for x:

$$\frac{2x}{L} < \frac{\mu_s}{3} \quad \Rightarrow \quad x < \frac{\mu_s}{6} L = L/42.$$

Therefore, the answer is A.

2018.21. The uniform bar is a physical pendulum. The formula (at small amplitudes) for the period of a physical pendulum with mass M, moment of inertia I about the axis of rotation, and distance x between the axis and center of mass is

$$T = 2\pi \sqrt{\frac{I}{Mgx}}.$$

In this case, using the parallel axis theorem, the moment of inertia I is

$$I = I_c + Mx^2 = \frac{1}{12} ML^2 + Mx^2.$$

Since g and 2π remain constant, to minimize T we wish to minimize I/Mx, which can be expressed in terms of x as

$$\frac{I}{Mx} = x + \left(\frac{1}{12} L^2\right) x^{-1}.$$

The right-hand side is in the famous form $c(y + 1/y)$, for a constant $c = L/2\sqrt{3}$ and $y = x/c$. Note that the sum of y and its reciprocal takes its minimum of 2 when $y = 1$, which can be proven either by taking the derivative or using classical inequalities[10]. Thus,

$$x = cy = c \cdot 1 = L/2\sqrt{3} \quad \Rightarrow \quad \text{the answer is B.}$$

[10] These include the AM-GM or Cauchy-Schwarz Inequalities. Alternatively, we can simply square the expression and re-factor to get $(y + 1/y)^2 = y^2 + 1/y^2 + 2 = y^2 + 1/y^2 - 2 + 4 = (y - 1/y)^2 + 4 \geq 4$, as desired.

2018.22. The pulley system turns tension from the string connected to the left mass, no matter what the direction, into an equal tension force acting directly upwards on the right mass. Thus the motion of the right mass is purely in the vertical axis, so we can rule out answer choice E. Furthermore, during the oscillation of the mass to the left, the tension changes, so the vertical motion of the mass to the right must also oscillate, ruling out answer choice A.

We now need to discern the net motion of the mass to the right, which can be done by comparing the average tension in one oscillation to mg. If there was no net vertical motion, then when the mass to the left oscillates once, the average vertical component of the tension is mg, and the horizontal component is nonzero. Therefore, the average magnitude of the tension should be greater than mg, which contradicts the assumption of no vertical motion. To correct this, the average tension in one oscillation must be greater than mg, so the right mass must move upward as it oscillates. Thus, the answer is B.

2018.23. The key to solving this problem is that the rod is *rigid*, meaning that the entire system of the two masses is a rigid body. Since no external forces are being applied to the rod, by Newton's First Law, its motion can be described as a constant translational velocity v_c, combined with a constant angular velocity ω about the center of mass.

We can calculate the velocity of the center of mass using the total momentum, which yields

$$p = (m_1 + m_2)v_c \quad \Rightarrow \quad v_c = \frac{m_1 v}{m_1 + m_2}.$$

Furthermore, the center of mass is a point at distances

$$r_1 = \frac{m_2}{m_1 + m_2}L \qquad \text{and} \qquad r_2 = \frac{m_1}{m_1 + m_2}L$$

from masses m_1 and m_2, respectively. We can use this to calculate the angular velocity as

$$v = v_c + r_1\omega = \frac{m_1 v}{m_1 + m_2} + \frac{m_2}{m_1 + m_2}L\omega \quad \Rightarrow \quad \omega = v/L.$$

Initially, the velocity of m_2 is zero, i.e., $\vec{v}_c + \vec{\omega} \times \vec{r}_2 = 0$. This means that $r_2\omega$ is equal in magnitude to v_c. The next time m_2 is at rest is when $\omega \times r_2$ is again in the opposite direction from v_c. This first occurs after one complete revolution, so the time is given by the period of rotation

$$t = \frac{2\pi}{\omega} = 2\pi L/v.$$

Thus, the answer is A.

2018.24. We present two different approaches here:

Approach 1: This approach involves a clever application of symmetry to solve the problem without using calculus. It was given as the intended solution for the problem.

First, by Newton's Third Law, the gravitational force on the particle from the shell is the same as the force on the shell from the particle. The key observation is that since the points on the shell are all symmetric and equidistant from the particle, the gravitational pull of the particle on each unit area of the shell is constant. If we reverse the direction of this force without changing its magnitude, we end up with a pressure of $P = Gm\sigma/R^2$ pushing the shell away from the particle.

We want to find the magnitude of the net force from this pressure. To do this, imagine an equivalent situation without the particle, instead putting a flat airtight lid on the hemisphere and filling it with gas of constant pressure P. The net force due to pressure of the gas on the closed hemisphere (with lid) is zero, as otherwise the hemisphere would accelerate on its own, which is absurd[11]. Thus, the force on the curved hemisphere is equal and opposite to the force on the lid. However, since the lid is flat, its net force can be easily computed as

$$F = PA = \frac{Gm\sigma}{R^2} \cdot \pi R^2 = \pi G m \sigma.$$

Thus, the answer is E.

Approach 2: This approach uses calculus. Although it requires more advanced mathematics than is tested on the $F = ma$, it is also more straightforward to come up with.

We can parameterize the points on the hemispherical surface with two angles, as follows. First, if we consider the radius connecting the center to the point, let the angle between this radius and the vertical be θ. Second, if we project the radius onto a horizontal plane, let φ be the angle it makes in that plane[12]. Then a mass element dm can be described as

$$dm = \sigma R^2 \sin\theta \, d\theta \, d\varphi.$$

Taking into account that the horizontal components of gravity cancel each

[11] You can also derive this fact mathematically by using the *divergence theorem*.

[12] The angles θ and φ are known as the *polar* and *azimuthal* angles in spherical coordinates.

other out for symmetry reasons, the net force on the particle is

$$\begin{aligned}
F &= \iint_S \frac{Gm}{R^2}\cos\theta\,dm \\
&= \int_0^{2\pi}\int_0^{\pi/2} Gm\sigma\sin\theta\cos\theta\,d\theta\,d\varphi \\
&= 2\pi Gm\sigma\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \\
&= 2\pi Gm\sigma\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\,d\theta \;=\; \pi Gm\sigma \quad\Rightarrow\quad \text{the answer is E.}
\end{aligned}$$

2018.25. We also present two approaches for this problem:

Approach 1: This approach uses approximation and significant figures to obtain the answer, and it is likely the approach a time-conscious student would take on the actual test.

The surface area of the wire is proportional to $r\ell$. The number of significant figures in the product of two measurements is approximately the minimum number of significant figures in either measurement. Thus, the initial measurement uncertainty is given by the ruler measurement, on the order of 10%. The other methods have estimated uncertainties of

1. $\approx 1\%$, as both measurements have three significant figures,
2. $\approx 10\%$, as the ruler is still the bottleneck in terms of uncertainty, and
3. $\approx 3.2\%$, as the uncertainty of a measurement is inversely proportional to $\sqrt{N}$, where N, the number of independent trials, is 10.

These approximations are sufficiently far apart from each other, so we can use them to conclude that the answer is D.

Approach 2: This approach uses calculation with relative uncertainties.

When two independent measurements are multiplied or divided, their relative uncertainties add geometrically. The relative uncertainties of each method of improving precision are then

1. $\sqrt{(1\%)^2 + (1\%)^2} \approx 1.4\%$,
2. $\sqrt{(10\%)^2 + (0.1\%)^2} \approx 10\%$, and
3. $\sqrt{(10\%)^2 + (1\%)^2}/\sqrt{10} \approx 3.3\%$.

Thus, the answer is D.

Note: We can see that the estimates from Approach 1 were quite close to the actual values, particularly for Methods 2 and 3, where the two measurements had very different relative uncertainties.

Useful "Stuff"

ToC

Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ε	epsilon	N	ν	nu	Φ	φ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Decimal prefixes

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{18}	exa-	E	10^{-1}	deci-	d
10^{15}	peta-	P	10^{-2}	centi-	c
10^{12}	tera-	T	10^{-3}	milli-	m
10^{9}	giga-	G	10^{-6}	micro-	μ
10^{6}	mega-	M	10^{-9}	nano-	n
10^{3}	kilo-	k	10^{-12}	pico-	p
10^{2}	hecto-	h	10^{-15}	femto-	f
10^{1}	deca-	da	10^{-18}	atto-	a

Rules for approximate calculations

Let $P = 3.14$ and $E = 2.718$ be approximations of π and e to 3 and 4 significant digits, respectively. Between them, the minimum number of significant digits is 3.

- Addition and subtraction: the result should have the minimum of the number of significant digits of the terms. For example,

$$P + E = 3.14 + 2.718 = 5.858 \approx 5.86.$$

- Multiplication and division: The numbers being multiplied are first rounded to the minimum number of significant digits, multiplied, and rounded to the minimum number of significant digits. For example,

$$P \cdot E = 3.14 \cdot 2.718 \approx 3.14 \cdot 2.72 = 8.5408 \approx 8.54.$$

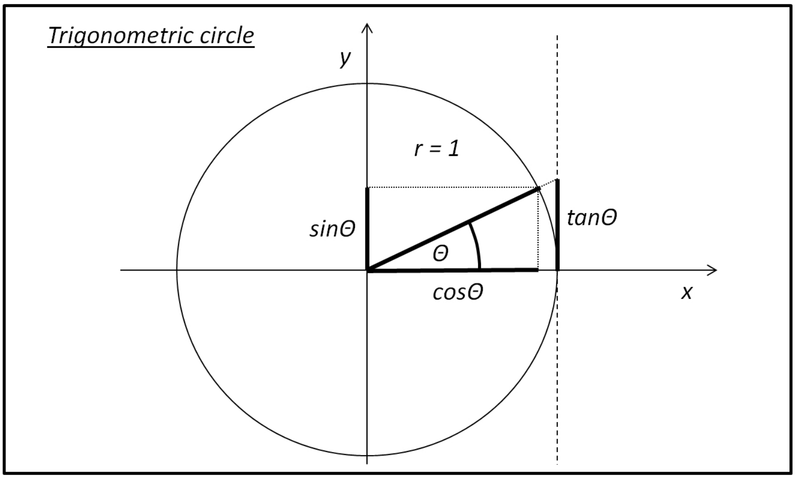
- Raising to the second or third power: The result should have the same number of significant digits as the base. For example,

$$P^2 = 3.14^2 = 9.8596 \approx 9.86.$$

- Extracting a square or cube root: The result should have the same number of significant digits as the radicand. For example,

$$\sqrt{E} = \sqrt{2.718} = 1.6486350 \approx 1.649.$$

Summary of Trigonometry



$\sin\theta$, $\cos\theta$, and $\tan\theta$ in the trigonometric circle

$$\boxed{-1 \le \sin\theta \le 1 \qquad -1 \le \cos\theta \le 1}$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \qquad \boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}} \qquad \boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$\boxed{\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}} \quad \boxed{\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}}$$

$$\boxed{\sin(x+y) = \sin x\cos y + \cos x\sin y} \quad \boxed{\cos(x+y) = \cos x\cos y - \sin x\sin y}$$

Summary of logarithms $(b > 0)$

$$\boxed{b^x = a \quad \Leftrightarrow \quad x = \log_b a}$$

$$\boxed{\log_b 1 = 0}$$

$$\boxed{\log x = \log_{10} x \qquad \ln x = \log_e x}$$

$$\boxed{\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}}$$

$$\boxed{\log_b xy = \log_b x + \log_b y} \qquad \boxed{\log_b \frac{x}{y} = \log_b x - \log_b y}$$

$$\boxed{\log_b x^y = y \log_b x}$$

Frequently used series and approximations (for $|x| \ll 1$)

$(1+x)^a$	$=$	$1 + ax + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \ldots$	$\approx$	$1 + ax$
e^x	$=$	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$	$\approx$	$1 + x$
$\ln(1+x)$	$=$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$	$\approx$	x
$\sin x$	$=$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$	$\approx$	x
$\cos x$	$=$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$	$\approx$	$1 - \dfrac{x^2}{2}$
$\tan x$	$=$	$x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \ldots$	$\approx$	x
$\arctan x$	$=$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$	$\approx$	x

Moments of inertia of simple shapes

point mass m at distance r from axis	$I = mr^2$
ring with mass m and radius r, axis perpendicular to ring plane through ring center	$I = mr^2$
disk with mass m and radius r, axis perpendicular to disk plane through disk center	$I = \frac{1}{2}mr^2$
hollow cylinder with mass m and radius r, axis perpendicular to cylinder base through center of base	$I = mr^2$
solid cylinder with mass m and radius r, axis perpendicular to cylinder base through center of base	$I = \frac{1}{2}mr^2$
hollow sphere with mass m and radius r, axis through center of sphere	$I = \frac{2}{3}mr^2$
solid ball with mass m and radius r, axis through center of sphere	$I = \frac{2}{5}mr^2$
stick of length L and mass m axis perpendicular to stick through center of stick	$I = \frac{1}{12}mL^2$
stick of length L and mass m axis perpendicular to stick through end of stick	$I = \frac{1}{3}mL^2$

Period of oscillations of various harmonic oscillators

m always denotes the mass of the pendulum,
I its moment of inertia, and g the gravitational acceleration

Simple pendulum ℓ is the length of the pendulum	$T = 2\pi\sqrt{\frac{\ell}{g}}$
Physical pendulum d is the pivot-centroid distance	$T = 2\pi\sqrt{\frac{I}{mgd}}$
Spring-mass (elastic) oscillator k is the spring constant	$T = 2\pi\sqrt{\frac{m}{k}}$
Torsional oscillator f is the torsional constant of the string	$T = 2\pi\sqrt{\frac{I}{f}}$
U-shaped tube with liquid ℓ is the total length of liquid	$T = 2\pi\sqrt{\frac{\ell}{2g}}$
A length-ℓ vertical stick with density ρ floating in liquid with density ρ_0	$T = 2\pi\sqrt{\frac{\rho\ell}{\rho_0 g}}$

Additional Resources

ToC

1. U.S. Physics Team's Web Site:
   https://www.aapt.org/physicsteam

2. David Morin,
   Problems and Solutions in Introductory Mechanics,
   CreateSpace Independent Publishing Platform, 2014.

3. David Morin,
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4. S. S. Krotov,
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   Mir Publishers, 1990.
   https://www.archive.org/details/ProblemsInPhysicsSSKrotov

5. V. S. Wolkenstein,
   Problems in General Physics,
   Mir Publishers, 1987.
   https://www.archive.org/details/WolkensteinProblemsInGeneralPhysicsMir

6. I. E. Irodov,
   Problems in General Physics,
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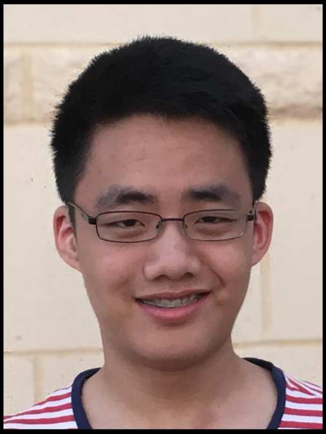
   https://www.archive.org/details/IrodovProblemsInGeneralPhysics

7. Branislav Kisačanin,
   Online Physics with Dr.Branislav,
   https://www.youtube.com/watch?v=lng7_rtpgq0&t=2s

8. AwesomeMath Academy Online Physics Courses:
   https://www.awesomemath.org/academy/online-courses



Dr. Branislav Kisačanin is a computer scientist at Nvidia Corporation. He is a highly regarded expert for applications of computer vision and artificial intelligence in autonomous driving. Dr. Branislav published five books on computer vision and control theory and was a guest editor for several special issues of top computer vision journals. In his spare time, he is a passionate teacher of competitive math and physics at the *AwesomeMath Summer Programs* and at the *AwesomeMath Academy*. In that arena, Dr. Branislav published three books on competitive math, with two more currently in the pipeline, and has created three levels of competitive physics courses at the *AwesomeMath Academy*. His physics students have won a number of medals at USAPhO competitions. This is his first book on competitive physics.

Some trivia about Dr. Branislav: He was born in a country that does not exist any more (Yugoslavia), and now lives in a ZIP code that is the 25^{th} Fibonacci number (75025).

Eric K. Zhang is a student at Plano West Senior High School, where he excels in math, physics, and computer science competitions. Eric has twice been a member of the US International Olympiad in Informatics team, and in 2018 he won a gold medal, placing 9th overall. Furthermore, after winning gold medals at the USAPhO in 2017 and 2018, he was invited to be a member of the U.S. Physics Team in those same years. Eric won an Honorable Mention at the 2017 USA Mathematical Olympiad and was a participant in the Mathematical Olympiad Program. He has previously taught competition math as an instructor at the *A* Winter Math Camp*. This is Eric's first book.

Some trivia about Eric: His current age (in years) is a Fermat prime, so a regular polygon with that many sides is constructible with straightedge and compass (17). Eric's ZIP code is also the 25^{th} Fibonacci number (75025).